PART A

Ordinary Differential Equations (ODEs)

Chap. 1 First-Order ODEs

Sec. 1.1 Basic Concepts. Modeling

To get a good start into this chapter and this section, quickly **review your basic calculus**. Take a look at the front matter of the textbook and see a review of the main differentiation and integration formulas. Also, Appendix 3, pp. A63–A66, has useful formulas for such functions as exponential function, logarithm, sine and cosine, etc. The beauty of ordinary differential equations is that the subject is quite systematic and has different methods for different types of ordinary differential equations, as you shall learn. Let us discuss some Examples of Sec. 1.1, pp. 4–7.

Example 2, p. 5. Solution by Calculus. Solution Curves. To solve the first-order ordinary differential equation (ODE)

$$y' = \cos x$$

means that we are looking for a function whose derivative is $\cos x$. Your first answer might be that the desired function is $\sin x$, because $(\sin x)' = \cos x$. But your answer would be incomplete because also $(\sin x + 2)' = \cos x$, since the derivative of 2 and of any constant is 0. Hence the complete answer is $y = \cos x + c$, where c is an arbitrary constant. As you vary the constants you get an infinite family of solutions. Some of these solutions are shown in **Fig. 3**. The lesson here is that you should never forget your constants!

Example 4, pp. 6–7. Initial Value Problem. In an initial value problem (IVP) for a first-order ODE we are given an ODE, here $y' = 3y$, and an initial value condition $y(0) = 5.7$. For such a problem, the first step is to solve the ODE. Here we obtain $y(x) = ce^{3x}$ as shown in **Example 3**, p. 5. Since we also have an initial condition, we must substitute that condition into our solution and get $y(0) = ce^{3 \cdot 0} = ce^0 = c \cdot 1 = c = 5.7$. Hence the complete solution is $y(x) = 5.7e^{3x}$. The lesson here is that for an initial value problem you get a unique solution, also known as a particular solution.

Modeling means that you interpret a physical problem, set up an appropriate mathematical model, and then try to solve the mathematical formula. Finally, you have to interpret your answer. Examples 3 (exponential growth, exponential decay) and 5 (radioactivity) are examples of modeling problems. Take a close look at **Example 5**, p. 7, because it outlines all the steps of modeling.

Problem Set 1.1. Page 8

3. Calculus. From Example 3, replacing the independent variable t by x we know that $y' = 0.2y$ has a solution $y = 0.2ce^{0.2x}$. Thus by analogy, $y' = y$ has a solution

$$1 \cdot ce^{1 \cdot x} = ce^x,$$

where c is an arbitrary constant.

Another approach (to be discussed in details in Sec. 1.3) is to write the ODE as

$$\frac{dy}{dx} = y,$$

and then by algebra obtain

$$dy = y\,dx, \quad \text{so that} \quad \frac{1}{y}dy = dx.$$

Integrate both sides, and then apply exponential functions on both sides to obtain the same solution as above

$$\int \frac{1}{y}dy = \int dx, \qquad \ln|y| = x + c, \qquad e^{\ln|y|} = e^{x+c}, \qquad y = e^x \cdot e^c = c^* e^x,$$
$$\text{(where } c^* = e^c \text{ is a constant).}$$

The technique used is called **separation of variables** because we separated the variables, so that y appeared on one side of the equation and x on the other side before we integrated.

7. Solve by integration. Integrating $y' = \cosh 5.13x$ we obtain (chain rule!) $y = \int \cosh 5.13x\,dx$ $= \frac{1}{5.13}(\sinh 5.13x) + c$. Check: Differentiate your answer:

$$\left(\tfrac{1}{5.13}(\sinh 5.13x) + c\right)' = \tfrac{1}{5.13}(\cosh 5.13x) \cdot 5.13 = \cosh 5.13x, \text{ which is correct.}$$

11. Initial value problem (IVP). (a) Differentiation of $y = (x + c)e^x$ by product rule and definition of y gives

$$y' = e^x + (x + c)e^x = e^x + y.$$

But this looks precisely like the given ODE $y' = e^x + y$. Hence we have shown that indeed $y = (x + c)e^x$ is a solution of the given ODE. **(b)** Substitute the initial value condition into the solution to give $y(0) = (0 + c)e^0 = c \cdot 1 = \frac{1}{2}$. Hence $c = \frac{1}{2}$ so that the answer to the IVP is

$$y = (x + \tfrac{1}{2})e^x.$$

(c) The graph intersects the x-axis at $x = 0.5$ and shoots exponentially upward.

19. Modeling: Free Fall. $y'' = g = \text{const}$ is the model of the problem, an ODE of second order. Integrate on both sides of the ODE with respect to t and obtain the velocity $v = y' = gt + c_1$ (c_1 arbitrary). Integrate once more to obtain the distance fallen $y = \frac{1}{2}gt^2 + c_1 t + c_2$ (c_2 arbitrary). To do these steps, we used calculus. From the last equation we obtain $y = \frac{1}{2}gt^2$ by imposing the initial conditions $y(0) = 0$ and $y'(0) = 0$, arising from the stone starting at rest at our choice of origin, that is the initial position is $y = 0$ with initial velocity 0. From this we have $y(0) = c_2 = 0$ and $v(0) = y'(0) = c_1 = 0$.

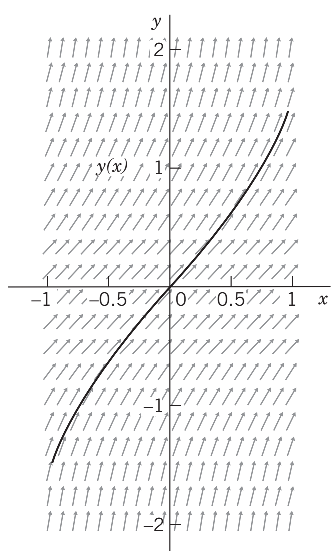
Sec. 1.2 Geometric Meaning of $y' = f(x, y)$. Direction Fields, Euler's Method

Problem Set 1.2. Page 11

1. Direction field, verification of solution. You may verify by differentiation that the general solution is $y = \tan(x + c)$ and the particular solution satisfying $y(\frac{1}{4}\pi) = 1$ is $y = \tan x$. Indeed, for the particular solution you obtain

$$y' = \frac{1}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = 1 + \tan^2 x = 1 + y^2$$

and for the general solution the corresponding formula with x replaced by $x + c$.



Sec. 1.2 Prob. 1. Direction Field

15. Initial value problem. Parachutist. In this section the usual notation is (1), that is, $y' = f(x, y)$, and the direction field lies in the xy-plane. In Prob. 15 the ODE is $v = f(t, v) = g - bv^2/m$, where v suggests velocity. Hence the direction field lies in the tv-plane. With $m = 1$ and $b = 1$ the ODE becomes $v' = g - v^2$. To find the limiting velocity we find the velocity for which the acceleration equals zero. This occurs when $g - v^2 = 9.80 - v^2 = 0$ or $v = 3.13$ (approximately). For $v < 3.13$ you have $v' > 0$ (increasing curves) and for $v > 3.13$ you have $v' < 0$ (decreasing curves). Note that the isoclines are the horizontal parallel straight lines $g - v^2 = \text{const}$, thus $v = \text{const}$.

Sec. 1.3 Separable ODEs. Modeling

Problem Set 1.3. Page 18

1. CAUTION! Constant of integration. It is important to introduce the constant of integration immediately, in order to avoid getting the wrong answer. For instance, let

$$y' = y. \qquad \text{Then} \qquad \ln|y| = x + c, \qquad y = c^* e^x \qquad (c^* = e^c),$$

which is the correct way to do it (the same as in Prob. 3 of Sec. 1.1 above) whereas introducing the constant of integration later yields

$$y' = y, \qquad \ln|y| = x, \qquad y = e^x + C$$

which is not a solution of $y' = y$ when $C \neq 0$.

5. General solution. Separating variables, we have $y\,dy = -36x\,dx$. By integration,

$$\tfrac{1}{2}y^2 = -18x^2 + \tilde{c}, \qquad y^2 = 2\tilde{c} - 36x^2, \qquad y = \pm\sqrt{c - 36x^2} \ \ (c = 2\tilde{c}).$$

With the plus sign of the square root we get the upper half and with the minus sign the lower half of the ellipses in the answer on p. A4 in Appendix 2 of the textbook.

For $y = 0$ (the x-axis) these ellipses have a vertical tangent, so that at points of the x-axis the derivative y' does not exist (is infinite).

17. Initial value problem. Using the extended method (8)–(10), let $u = y/x$. Then by product rule $y' = u + xu'$. Now

$$y' = \frac{y + 3x^4\cos^2(y/x)}{x} = \frac{y}{x} + 3x^3\cos\left(\frac{y}{x}\right) = u + 3x^3\cos^2 u = u + x(3x^2\cos^2 u)$$

so that $\quad u' = 3x^2\cos^2 u$.

Separating variables, the last equation becomes

$$\frac{du}{\cos^2 u} = 3x^2 dx.$$

Integrate both sides, on the left with respect to u and on the right with respect to x, as justified in the text then solve for u and express the intermediate result in terms of x and y

$$\tan u = x^3 + c, \qquad u = \frac{y}{x} = \arctan(x^3 + c), \qquad y = xu = x\arctan(x^3 + c).$$

Substituting the initial condition into the last equation, we have

$$y(1) = 1\arctan(1^3 + c) = 0, \quad \text{hence} \quad c = -1.$$

Together we obtain the answer

$$y = x\arctan(x^3 - 1).$$

23. Modeling. Boyle–Mariotte's law for ideal gases. From the given information on the rate of change of the volume

$$\frac{dV}{dP} = -\frac{V}{P}.$$

Separating variables and integrating gives

$$\frac{dV}{V} = -\frac{dP}{P}, \qquad \int \frac{1}{V}dV = -\int \frac{1}{P}dP, \qquad \ln|V| = -\ln|P| + c.$$

Applying exponents to both sides and simplifying

$$e^{\ln|V|} = e^{-\ln|P|+c} = e^{-\ln|P|} \cdot e^c = \frac{1}{e^{\ln|P|}} \cdot e^c = \frac{1}{|P|}e^c.$$

Hence we obtain for nonnegative V and P the desired law (with $c^* = e^c$, a constant)

$$V \cdot P = c^*.$$

Sec. 1.4 Exact ODEs. Integrating Factors

Use (6) or (6*), on p. 22, only if inspection fails. Use only one of the two formulas, namely, that in which the integration is simpler. For integrating factors try both Theorems 1 and 2, on p. 25. Usually only one of them (or sometimes neither) will work. There is no completely systematic method for integrating factors, but these two theorems will help in many cases. Thus this section is slightly more difficult.

Problem Set 1.4. Page 26

1. Exact ODE. We proceed as in Example 1 of Sec. 1.4. We can write the given ODE as

$$M\,dx + N\,dy = 0 \qquad \text{where } M = 2xy \text{ and } N = x^2.$$

Next we compute $\dfrac{\partial M}{\partial y} = 2x$ (where, when taking this partial derivative, we treat x as if it were a constant) and $\dfrac{\partial N}{\partial x} = 2x$ (we treat y as if it were a constant). (See Appendix A3.2 for a review of partial derivatives.) This shows that the ODE is exact by (5) of Sec. 1.4. From (6) we obtain by integration

$$u = \int M\,dx + k(y) = \int 2xy\,dx + k(y) = x^2y + k(y).$$

To find $k(y)$ we differentiate this formula with respect to y and use (4b) to obtain

$$\frac{\partial u}{\partial y} = x^2 + \frac{dk}{dy} = N = x^2.$$

From this we see that

$$\frac{dk}{dy} = 0, \qquad k = \text{const.}$$

The last equation was obtained by integration. Insert this into the equation for u, compare with (3) of Sec. 1.4, and obtain $u = x^2y + c^*$. Because u is a constant, we have

$$x^2y = c, \qquad \text{hence} \qquad y = c/x^2.$$

5. Nonexact ODE. From the ODE, we see that $P = x^2 + y^2$ and $Q = 2xy$. Taking the partials we have $\dfrac{\partial P}{\partial y} = 2y$ and $\dfrac{\partial Q}{\partial x} = -2y$ and, since they are not equal to each other, the ODE is nonexact. Trying Theorem 1, p. 25, we have

$$R = \frac{(\partial P/\partial y - \partial Q/\partial x)}{Q} = \frac{2y + 2y}{-2xy} = \frac{4y}{-2xy} = -\frac{2}{x}$$

which is a function of x only so, by (17), we have $F(x) = \exp \int R(x)\,dx$. Now

$$\int R(x)\,dx = -2\int \frac{1}{x}dx = -2\ln x = \ln(x^{-2}) \qquad \text{so that} \qquad F(x) = x^{-2}.$$

Then

$$M = FP = 1 + x^{-2}y^2 \quad \text{and} \quad N = FQ = -2x^{-1}y. \quad \text{Thus} \quad \frac{\partial M}{\partial y} = 2x^{-2}y = \frac{\partial N}{\partial x}.$$

This shows that multiplying by our integrating factor produced an exact ODE. We solve this equation using 4(b), p. 21. We have

$$u = \int -2x^{-1}y\,dy = -2x^{-1}\int y\,dy = -x^{-1}y^2 + k(x).$$

From this we obtain

$$\frac{\partial u}{\partial x} = x^{-2}y^2 + \frac{dk}{dx} = M = 1 + x^{-2}y^2, \quad \text{so that} \quad \frac{dk}{dx} = 1 \quad \text{and} \quad k = \int dx = x + c^*.$$

Putting k into the equation for u, we obtain

$$u(x,y) = -x^{-1}y^2 + x + c^* \qquad \text{and putting it in the form of (3)} \qquad u = -x^{-1}y^2 + x = c.$$

Solving explicitly for y requires that we multiply both sides of the last equation by x, thereby obtaining (with our constant $= -$constant on p. A5)

$$-y^2 + x^2 = cx, \qquad y = (x^2 - cx)^{1/2}.$$

9. Initial value problem. In this section we usually obtain an implicit rather than an explicit general solution. The point of this problem is to illustrate that in solving initial value problems, one can proceed directly with the implicit solution rather than first converting it to explicit form.

The given ODE is exact because (5) gives

$$M_y = \frac{\partial}{\partial y}(2e^{2x}\cos y) = -2e^{2x}\sin y = N_x.$$

From this and (6) we obtain, by integration,

$$u = \int M\,dx = \int 2\,e^{2x}\cos y\,dx = e^{2x}\cos y + k(y).$$

$u_y = N$ now gives

$$u_y = -e^{2x}\sin y + k'(y) = N, \qquad k'(y) = 0, \qquad k(y) = c^* = \text{const.}$$

Hence an implicit general solution is

$$u = e^{2x} \cos y = c.$$

To obtain the desired particular solution (the solution of the initial value problem), simply insert $x = 0$ and $y = 0$ into the general solution obtained:

$$e^0 \cos 0 = 1 \cdot 1 = c.$$

Hence $c = 1$ and the answer is

$$e^{2x} \cos y = 1.$$

This implies

$$\cos y = e^{-2x}, \qquad \text{thus the explicit form} \qquad y = \arccos(e^{-2x}).$$

15. Exactness. We have $M = ax + by$, $N = kx + ly$. The answer follows from the exactness condition (5), p. 21. The calculation is

$$M_y = b = N_x = k, \qquad M = ax + ky, \qquad u = \int M\, dx = \frac{1}{2}ax^2 + kxy + \kappa(y)$$

with $\kappa(y)$ to be determined from the condition

$$u_y = kx + \kappa'(y) = N = kx + ly, \qquad \text{hence} \qquad \kappa' = ly.$$

Integration gives $\kappa = \frac{1}{2}ly^2$. With this κ, the function u becomes

$$u = \tfrac{1}{2}ax^2 + kxy + \tfrac{1}{2}ly^2 = \text{const.}$$

(If we multiply the equation by a factor 2, for beauty, we obtain the answer on p. A5).

Sec. 1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

Example 3, pp. 30–31. Hormone level. The integral

$$I = \int e^{Kt} \cos \frac{\pi t}{12}\, dt$$

can be evaluated by integration by parts, as is shown in calculus, or, more simply, by undetermined coefficients, as follows. We start from

$$\int e^{Kt} \cos \frac{\pi t}{12} dt = e^{Kt}\left(a \cos \frac{\pi t}{12} + b \sin \frac{\pi t}{12}\right)$$

with a and b to be determined. Differentiation on both sides and division by e^{Kt} gives

$$\cos \frac{\pi t}{12} = K\left(a \cos \frac{\pi t}{12} + b \sin \frac{\pi t}{12}\right) - \frac{a\pi}{12} \sin \frac{\pi t}{12} + \frac{b\pi}{12} \cos \frac{\pi t}{12}.$$

We now equate the coefficients of sine and cosine on both sides. The sine terms give

$$0 = Kb - \frac{a\pi}{12}, \qquad \text{hence} \qquad a = \frac{12K}{\pi} b.$$

The cosine terms give

$$1 = Ka + \frac{\pi}{12}b = \left(\frac{12K^2}{\pi} + \frac{\pi}{12}\right)b = \frac{144K^2 + \pi^2}{12\pi}b.$$

Hence,

$$b = \frac{12\pi}{144K^2 + \pi^2}, \qquad a = \frac{144K}{144K^2 + \pi^2}.$$

From this we see that the integral has the value

$$e^{Kt}\left(a\cos\frac{\pi t}{12} + b\sin\frac{\pi t}{12}\right) = \frac{12\pi}{144K^2 + \pi^2}e^{Kt}\left(\frac{12K}{\pi}\cos\frac{\pi t}{12} + \sin\frac{\pi t}{12}\right).$$

This value times B (a factor we did not carry along) times e^{-Kt} (the factor in front of the integral on p. 31) is the value of the second term of the general solution and of the particular solution in the example.

Example 4, pp. 32–33. Logistic equation, Verhulst equation. This ODE

$$y' = Ay - By^2 = Ay\left(1 - \frac{B}{A}y\right)$$

is a basic population model. In contrast to the Malthus equation $y' = ky$, which for a positive initial population models a population that grows to infinity (if $k > 0$) or to zero (if $k < 0$), the logistic equation models growth of small initial populations and decreasing populations of large initial populations. You can see directly from the ODE that the dividing line between the two cases is $y = A/B$ because for this value the derivative y' is zero.

Problem Set 1.5. Page 34

5. Linear ODE. Multiplying the given ODE (with $k \neq 0$) by e^{kx}, you obtain

$$(y' + ky)e^{kx} = e^{-kx}e^{ks} = e^0 = 1.$$

The left-hand side of our equation is equal to $(ye^{kx})'$, so that we have

$$(ye^{kx})' = 1.$$

Integration on both sides gives the final answer.

$$ye^{kx} = x + c, \qquad y = (x + c)e^{-kx}.$$

The use of (4), p. 28, is simple, too, namely, $p(x) = k, h = \int p(x)\,dx = \int k\,dx = kx$. Furthermore, $r = e^{-kx}$. This gives

$$\begin{aligned} y &= e^{-kx}\left(\int e^{kx}e^{-kx}dx + c\right) \\ &= e^{-kx}\left(\int 1\,dx + c\right) = e^{-kx}(x + c). \end{aligned}$$

9. Initial value problem. For the given ODE $y' + y \sin x = e^{\cos x}$ we have in (4)

$$p(x) = \sin x$$

so that by integration

$$h = \int \sin x \, dx = -\cos x$$

Furthermore the right-hand side of the ODE $r = e^{\cos x}$. Evaluating (4) gives us the general solution of the ODE. Thus

$$\begin{aligned} y &= e^{\cos x}\left(\int e^{-\cos x} \cdot e^{\cos x} \, dx + c\right) \\ &= e^{\cos x}(x + c). \end{aligned}$$

We turn to the initial condition and substitute it into our general solution and obtain the value for c

$$y(0) = e^{\cos 0}(0 + c) = -2.5, \qquad c = -\frac{2.5}{e}$$

Together the final solution to the IVP is

$$y = e^{\cos x}\left(x - \frac{2.5}{e}\right).$$

23. Bernoulli equation. In this ODE $y' + xy = xy^{-1}$ we have $p(x) = x$, $g(x) = x$ and $a = -1$. The new dependent variable is $u(x) = [y(x)]^{1-a} = y^2$. The resulting linear ODE (10) is

$$u' + 2xu = 2x.$$

To this ODE we apply (4) with $p(x) = 2x$, $r(x) = 2x$ hence

$$h = \int 2x \, dx = x^2, \qquad -h = -x^2$$

so that (4) takes the form

$$u = e^{-x^2}\left(\int e^{x^2}(2x) \, dx + c\right).$$

In the integrand, we notice that $(e^{x^2})' = (e^{x^2}) \cdot 2x$, so that the equation simplifies to

$$u = e^{-x^2}(e^{x^2} + c) = 1 + ce^{-x^2}.$$

Finally, $u(x) = y^2$ so that $y^2 = 1 + ce^{-x^2}$. From the initial condition $[y(0)]^2 = 1 + c = 3^2$. It follows that $c = 8$. The final answer is

$$y = 1 + 8e^{-x^2}.$$

31. Newton's law of cooling. Take a look at Example 6 in Sec. 1.3, pp. 15–16. Newton's law of cooling is given by

$$\frac{dT}{dt} = K(T - T_A).$$

In terms of the given problem, Newton's law of cooling means that the rate of change of the temperature T of the cake at any time t is proportional to the difference of temperature of the cake and the temperature T_A of the room. Example 6 also solves the equation by separation of variables and arrives at

$$T(t) = T_A + ce^{kt}.$$

At time $t = 0$, we have $T(0) = 300 = 60 + c \cdot e^{0 \cdot k} = 60 + c$, which gives that $c = 240$. Insert this into the previous equation with $T_A = 60$ and obtain

$$T(t) = 60 + 240e^{kt}.$$

Ten minutes later is $t = 10$ and we know that the cake has temperature $T(10) = 200$ [°F]. Putting this into the previous equation we have

$$T(10) = 60 + 240e^{10k} = 200, \quad e^k = \left(\tfrac{7}{12}\right)^{1/10}, \qquad k = \tfrac{1}{10}\ln\left(\tfrac{7}{12}\right) = -0.0539.$$

Now we can find out the time t when the cake has temperature of $T(t) = 61$°F. We set up, using the computed value of k from the previous step,

$$60 + 240e^{-0.0539t} = 61, \quad e^{-0.0539t} = \frac{1}{240}, \quad t = \frac{-\ln(240)}{-0.0539} = \frac{-5.48}{-0.0539} = 102\,\text{min}.$$

Sec. 1.6 Orthogonal Trajectories

The method is rather general because one-parameter families of curves can often be represented as general solutions of an ODE of first order. Then replacing $y' = f(x, y)$ by $\tilde{y}' = -1/f(x, \tilde{y})$ gives the ODE of the trajectories to be solved because two curves intersect at a right angle if the product of their slopes at the point of intersection equals -1; in the present case, $y'\tilde{y}' = -1$.

Problem Set 1.6. Page 38

9. Orthogonal trajectories. Bell-shaped curves. Note that the given curves $y = ce^{-x^2}$ are bell-shaped curves centered around the y-axis with the maximum value $(0, c)$ and tangentially approaching the x-axis for increasing $|x|$. For negative c you get the bell-shaped curves reflected about the x-axis. Sketch some of them. The first step in determining orthogonal trajectories usually is to solve the given representation $G(x, y, c) = 0$ of a family of curves for the parameter c. In the present case, $ye^{x^2} = c$. Differentiation with respect to x then gives (chain rule!)

$$y'e^{x^2} + 2xye^{x^2} = 0, \qquad y' + 2xy = 0.$$

where the second equation results from dividing the first by e^{x^2}.

Hence the ODE of the given curves is $y' = -2xy$. Consequently, the trajectories have the ODE $\tilde{y}' = 1/(2x\tilde{y})$. Separating variables gives

$$2\tilde{y}\,d\tilde{y} = dx/x. \quad \text{By integration,} \quad 2\tilde{y}^2/2 = -\ln|x| + c_1, \quad \tilde{y}^2 = -\ln|x| + c_1.$$

Taking exponents gives

$$e^{\tilde{y}^2} = x \cdot c_2. \qquad \text{Thus,} \qquad x = \tilde{c}e^{\tilde{y}^2}$$

where the last equation was obtained by letting $\tilde{c} = 1/c_2$. These are curves that pass through $(\tilde{c}, 0)$ and grow extremely rapidly in the positive x direction for positive $\tilde{c}$ with the x-axis serving as an axis of symmetry. For negative $\tilde{c}$ the curves open sideways in the negative x direction. Sketch some of them for positive and negative $\tilde{c}$ and see for yourself.

12. Electric field. To obtain an ODE for the given curves (circles), you must get rid of c. For this, multiply $(y-c)^2$ out. Then a term c^2 drops out on both sides and you can solve the resulting equation algebraically for c. The next step then is differentiation of the equation just obtained.

13. Temperature field. The given temperature field consists of upper halfs of ellipses (i.e., they do not drop below the x-axis). We write the given equation as

$$G(x, y, c) = 4x^2 + 9y^2 - c = 0 \qquad y > 0.$$

Implicit differentiation with respect to x, using the chain rule, yields

$$8x + 18yy' = 0 \qquad \text{and} \qquad y' = -\frac{4x}{9y}.$$

Using (3) of Sec. 1.6, we get

$$\tilde{y}' = -\frac{1}{4x/9\tilde{y}} = \frac{9\tilde{y}}{4x} \qquad \text{so that} \qquad \frac{d\tilde{y}}{dx} = \frac{9\tilde{y}}{4x} \qquad \text{and} \qquad d\tilde{y}\frac{1}{9\tilde{y}} = dx\frac{1}{4x}.$$

Integrating both sides gives

$$\frac{1}{9}\int \frac{1}{\tilde{y}}d\tilde{y} = \frac{1}{4}\int \frac{1}{x}dx \qquad \text{and} \qquad \frac{1}{9}\ln|\tilde{y}| = \frac{1}{4}\ln|x| + c_1.$$

Applying exponentiation on both sides and using (1) of Appendix 3, p. A63, gives the desired result $y = x^{9/4} \cdot \tilde{c}$, as on p. A5. The curves all go through the origin, stay above the x-axis, and are symmetric to the y-axis.

Sec. 1.7 Existence and Uniqueness of Solutions for Initial Value Problems

Since absolute values are always nonnegative, the only solution of $|y'| + |y| = 0$ is $y = 0$ ($y(x) \equiv 0$ for all x) and this function cannot satisfy the initial condition $y(0) = 1$ or any initial condition $y(0) = y_0$ with $y_0 \neq 0$.

The next ODE in the text $y' = 2x$ has the general solution $y = x^2 + c$ (calculus!), so that $y(0) = c = 1$ for the given initial condition.

The third ODE $xy' = y - 1$ is separable,

$$\frac{dy}{y-1} = \frac{dx}{x}.$$

By integration,

$$\ln|y-1| = \ln|x| + c_1, \qquad y - 1 = cx, \qquad y = 1 + cx,$$

a general solution which satisfies $y(0) = 1$ with any c because c drops out when $x = 0$. This happens only at $x = 0$. Writing the ODE in standard form, with y' as the first term, you see that

$$y' - \frac{1}{x}y = -\frac{1}{x},$$

showing that the coefficient $1/x$ of y is infinite at $x = 0$.

Theorems 1 and 2, pp. 39–40, concern initial value problems

$$y' = f(x, y), \qquad y(x) = y_0.$$

It is good to remember the two main facts:

1. Continuity of $f(x, y)$ is enough to guarantee the existence of a solution of (1), but is not enough for uniqueness (as is shown in Example 2 on p. 42).
2. Continuity of f and of its partial derivative with respect to y is enough to have uniqueness of the solution of (1), p. 39.

Problem Set 1.7. Page 42

1. Linear ODE. In this case the solution is given by the integral formula (4) in Sec. 1.5, which replaces the problem of solving an ODE by the simpler task of evaluating integrals – this is the point of (4). Accordingly, we need only conditions under which the integrals in (4) exist. The continuity of f and r are sufficient in this case.

3. Vertical strip as "rectangle." In this case, since a is the smaller of the numbers a and b/K and K is constant and b is no longer restricted, the answer $|x - x_0| < a$ given on p. A6 follows.

Chap. 2 Second-Order Linear ODEs

Chapter 2 presents different types of second-order ODEs and the specific techniques on how to solve them. The methods are systematic, but it requires practice to be able to identify with what kind of ODE you are dealing (e.g., a homogeneous ODE with constant coefficient in Sec. 2.2 or an Euler–Cauchy equation in Sec. 2.5, or others) and to recall the solution technique. However, you should know that there are only a few ideas and techniques and they are used repeatedly throughout the chapter. More theoretical sections are interspersed with sections dedicated to modeling applications (e.g., forced oscillations, resonance in Sec. 2.8, electric circuits in Sec. 2.9). The bonus is that, if you understand the methods of solving second-order linear ODEs, then you will have no problem in solving such ODEs of higher order in Chap. 3.

Sec. 2.1 Homogeneous Linear ODEs of Second Order

Take a look at pp. 46–47. Here we extend concepts defined in Chap. 1 for *first-order* ODEs, notably solution and homogeneous and nonhomogeneous, to *second-order* ODEs. To see this, look into Secs. 1.1 and 1.5 before we continue.

We will see in this section that a *homogeneous linear* ODE is of the form

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

An initial value problem for it will consist of *two* conditions, prescribing an *initial value* and an *initial slope* of the solution, both at the same point x_0. But, on the other hand, a general solution will now involve *two* arbitrary constants for which some values can be determined from the two initial conditions. Indeed, a general solution is of the form

$$y = c_1 y_1 + c_2 y_2 \tag{5}$$

where y_1 and y_2 are such that they cannot be pooled together with just one arbitrary constant remaining. The technical term for this is *linear independence.* We call y_1 and y_2 "linearly independent," meaning that they are not proportional on the interval on which a solution of the initial value problem is sought.

Problem Set 2.1. Page 53

As noted in Probs. 1 and 2, there are two cases of reduction of order that we wish to consider: **Case A:** x does not appear explicitly and **Case B:** y does not appear explicitly. The most general second-order ODE is of the form $F(x, y, y', y'') = 0$. The method of solution starts the same way in both cases. They can be reduced to first order by setting $z = y' = dy/dx$. With this change of variable and the chain rule,

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dy} \cdot \frac{dy}{dx} = \frac{dz}{dy} z.$$

The third equality is obtained by noting that, when we substitute $y' = z$ into the term dy'/dy, we get $\boxed{dy'/dy = dz/dy}$. Furthermore, $y' = dy/dx = z$, so that together $y'' = dy'/dy \cdot dy/dx = (dz/dy)z$.

3. Reduction to first order. Case B: *y* does not appear explicitly. The ODE $y'' + y' = 0$ is Case B, so, from the above, set $z = y' = dy/dx$, to give

$$\frac{dz}{dy} z = -z.$$

Separation of variables (divide by z) and integrating gives

$$\int z\,dz = -\int dy, \quad \text{thus } z = -y + c_1.$$

But $z = y'$, so our separation of variables gives us the following linear ODE:

$$y' + y = c_1.$$

We can solve this ODE by (4) of Sec. 1.5 with $p = 1$ and $r = c_1$. Then

$$h = \int p\,dx = \int dx = x \qquad \text{and} \qquad y(x) = e^{-x}\left(\int e^x c_1 dx + c_2\right) = e^{-x}(c_1 e^x + c_2),$$

which simplifies to the answer given on p. A6 (except for what we called the constants). It is important to remember the trick of reducing the second derivative by setting $z = y'$.

7. Reduction to first order. Case A: x does not appear explicitly. The ODE $y'' + y'^3 \sin y = 0$ is Case A, as explained above. After writing the ODE as $y'' = -\,y'^3 \sin y$, we reduce it, by setting $z = y' = dy/dx$, to give

$$y'' = \frac{dz}{dy} z = -z^3 \sin y.$$

Division by $-\,z^3$ (in the second equation) and separation of variables yields

$$-\frac{dz}{dy}\frac{1}{z^2} = \sin y, \qquad -\frac{dz}{z^2} = \sin y\,dy.$$

Integration gives

$$\frac{1}{z} = -\cos y + c_1.$$

Next use $z = dy/dx$, hence $1/z = dx/dy$, and separate again, obtaining $dx = (-\cos\, y + c_1)\,dy$. By integration, $x = -\sin y + c_1 y + c_2$. This is given on p. A6. The derivation shows that the two arbitrary constants result from the two integrations, which finally gave us the answer. Again, the trick of reducing the second derivative should be remembered.

17. General solution. Initial value problem. Just substitute $x^{3/2}$ and $x^{-1/2}$ ($x \neq 0$) and see that the two given solutions satisfy the given ODE. (The simple algebraic derivation of such solutions will be shown in Sec. 2.5 starting on p. 71.) They are linearly independent (not proportional) on any interval not containing 0 (where $x^{-1/2}$ is not defined). Hence $y = c_1 x^{3/2} + c_2\,x^{-1/2}$ is a general solution of the given ODE. Set $x = 1$ and use the initial conditions for y and $y' = \frac{3}{2}c_1 x^{1/2} - \frac{1}{2}c_2 x^{-3/2}$, where the equation for y' was obtained by differentiation of the general solution. This gives

(a) $$y(1) = \ c_1 + \ c_2 = -3$$

(b) $$y'(1) = \tfrac{3}{2}c_1 - \tfrac{1}{2}c_2 = 0.$$

We now use elimination to solve the system of two linear equations (a) and (b). Multiplying (b) by 2 and solving gives (c) $c_2 = 3c_1$. Substituting (c) in (a) gives $c_1 + 3c_1 = -3$ so that (d) $\boxed{c_1 = -\frac{3}{4}}$. Substituting (d) in (a) gives $\boxed{c_2 = -\frac{9}{4}}$. Hence the solution of the given initial value problem (IVP) is

$$y = -0.75x^{3/2} - 2.25x^{-1/2}.$$

Sec. 2.2 Homogeneous Linear ODEs with Constant Coefficients

To solve such an ODE

$$y'' + a\,y' + b\,y = 0 \qquad (a, b \text{ constant}) \tag{1}$$

amounts to first solving the quadratic equation

$$\lambda^2 + a\lambda + b = 0 \tag{3}$$

and identifying its roots. From algebra, we know that (3) may have *two real roots* λ_1, λ_2, *a real double root* λ, or *complex conjugate roots* $-\frac{1}{2}a + i\omega$, $-\frac{1}{2}a - i\omega$ with $i = \sqrt{-1}$ and $\omega = \sqrt{b - \frac{1}{4}a^2}$. Then the type of root (Case I, II, or III) determines the general solution to (1). Case I gives (6), Case II gives (7), and Case III gives (9). You may want to use the convenient table "Summary of Cases I–III" on p. 58. In (9) we have oscillations, harmonic if $a = 0$ and damped (going to zero as x increases) if $a > 0$. See Fig. 32 on p. 57 of the text.

The key in the derivation of (9), p. 57, is the Euler formula (11), p. 58, with $t = \omega x$, that is,

$$e^{i\omega x} = \cos\omega x + i\sin\omega x$$

which we will also need later.

Problem Set 2.2. Page 59

13. General solution. Real double root. Problems 1–15 amount to solving a quadratic equation. Observe that (3) and (4) refer to the "standard form," namely, the case that y'' has the coefficient 1. Hence we have to divide the given ODE $9y'' - 30y' + 25y = 0$, by 9, so that the given ODE in **standard form** is

$$y'' - \tfrac{30}{9}y' + \tfrac{25}{9}y = 0.$$

The corresponding characteristic equation is

$$\lambda^2 - \tfrac{30}{9}\lambda + \tfrac{25}{9} = 0.$$

From elementary algebra we know that the roots of *any* quadratic equation $ax^2 + bx + c = 0$ are $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, so that here we have, if we use this formula [instead of (4) where a and b are used in a different way!],

$$\lambda_{1,2} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{\frac{30}{9} \pm \sqrt{\left(\frac{30}{9}\right)^2 - 4 \cdot 1 \cdot \frac{25}{9}}}{2} = \frac{\frac{30}{9} \pm \sqrt{\frac{900}{81} - \frac{900}{81}}}{2} = \frac{5}{3}.$$

(The reason we used this formula here instead of (4) of Sec. 2.2 is that it is most likely familiar to you, that is, we assume you had to memorize it at some point.) Thus we see that the characteristic equation factors

$$\lambda^2 - \tfrac{30}{9}\lambda + \tfrac{25}{9} = \left(\lambda - \tfrac{5}{3}\right)^2 = 0.$$

This shows that it has a real double root (**Case II**) $\lambda = \frac{5}{3}$. Hence, as in Example 3 on p. 56 (or use "Summary of Cases I–III" on p. 58), the ODE has the general solution

$$y = (c_1 + c_2 x)e^{5x/3}.$$

15. Complex roots. The ODE $y'' + 0.54y' + (0.0729 + \pi)y = 0$ has the characteristic equation $\lambda^2 + 0.54\lambda + (0.0729 + \pi) = 0$, whose solutions are (noting that $\sqrt{-4\pi} = \sqrt{-1}\sqrt{4\pi} = i \cdot 2\sqrt{\pi}$)

$$\lambda_{1,2} = \frac{-0.54 \pm \sqrt{(0.54)^2 - 4 \cdot (0.0729 + \pi)}}{2}$$
$$= \frac{-0.54 \pm \sqrt{0.2916 - 0.2916 - 4\pi}}{2} = -0.27 \pm i\sqrt{\pi}.$$

This gives the real general solution (see Example 5 on p. 57 or **Case III** in the table "Summary of Cases I–III" on p. 58).

$$y = e^{-0.27x}(A\cos(\sqrt{\pi}x) + B\sin(\sqrt{\pi}x)).$$

This represents oscillations with a decreasing amplitude. See Graph in Prob. 29.

29. Initial value problem. We continue by looking at the solution of Prob. 15. We have additional information, that is, two initial conditions $y(0) = 0$ and $y'(0) = 1$. The first initial condition we can substitute immediately into the general solution and obtain

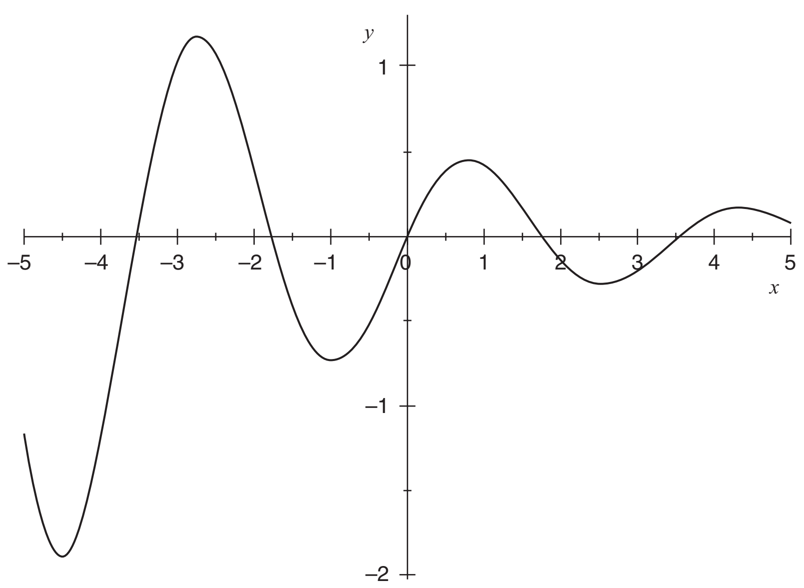
$$y(0) = e^{-0.27 \cdot 0}(A\cos(\sqrt{\pi}0) + B\sin(\sqrt{\pi}0)) = 1 \cdot A = 0, \quad \text{thus } A = 0.$$

The second initial condition concerns y'. We thus have to compute y' first, from our general solution (with $A = 0$). Using product and chain rules we have

$$y' = -0.27e^{-0.27x}(B\sin(\sqrt{\pi}x)) + e^{-0.27x}(\sqrt{\pi}B(\cos(\sqrt{\pi}x)).$$

Subsituting $x = 0$ into the equation for y' we get $y'(0) = 0 + 1 \cdot (\sqrt{\pi} \cdot B) = 1$ by the second initial condition. Hence $B = 1/\sqrt{\pi}$. Together, substituting $A = 0$ and $B = 1/\sqrt{\pi}$ into the formula for the general solution, gives

$$y = e^{-0.27x}(0 \cdot \cos(\sqrt{\pi}x) + \frac{1}{\sqrt{\pi}}\sin(\sqrt{\pi}x)) = \frac{1}{\sqrt{\pi}}e^{-0.27x}\sin(\sqrt{\pi}x)$$



Sec. 2.2 Prob. 29. Graph of IVP. Oscillations with a Decreasing Amplitude

31. Linear independence. This follows by noting that e^{kx} and xe^{kx} correspond to a basis of a homogenous ODE (1) as given in Case II of table "Summary of Cases I–III." Being a basis means, by definition of basis, that they are independent. The corresponding ODE is $y'' - 2ky' + k^2y = 0$. [We start with $e^{-ax/2} = e^{kx}$ so that $a = -2k$. The double root is $\lambda = -\frac{1}{2}a = -\frac{1}{2}(-2k) = k$. This determines our characteristic equation and finally the ODE.]

35. Linear dependence. This follows by noting that $\sin 2x = 2 \sin x \cos x$. Thus the two functions are linearly dependent. The problem is typical of cases in which some functional relation is used to show linear dependence, as discussed on p. 50 in Sec. 2.1.

Sec. 2.3 Differential Operators

Problem Set 2.3. Page 61

3. Differential operators. For e^{2x} we obtain $(D - 2I)e^{2x} = 2e^{2x} - 2e^{2x} = 0$. Since $(D - 2I)e^{2x} = 0$, applying $(D - 2I)$ twice to e^{2x} will also be 0, i.e., $(D - 2I)^2 e^{2x} = 0$. For xe^{2x} we first have

$$(D - 2I)xe^{2x} = Dxe^{2x} - 2Ixe^{2x} = e^{2x} + 2xe^{2x} - 2xe^{2x} = e^{2x}.$$

Hence $(D - 2I)xe^{2x} = e^{2x}$. Applying $D - 2I$ again, the right side gives

$$(D - 2I)^2 xe^{2x} = (D - 2I)e^{2x} = 2e^{2x} - 2e^{2x} = 0.$$

Hence xe^{2x} is a solution only in the case of a real double root (see the table on p. 58, Case II), the ODE being

$$(D - 2I)^2 y = (D^2 - 4D + 4I)y = y'' - 4y' + 4y = 0.$$

For e^{-2x} we obtain

$$(D - 2I)^2 e^{-2x} = (D^2 - 4D + 4I)e^{-2x} = 4e^{-2x} + 8e^{-2x} + 4e^{-2x} = 16e^{-2x}.$$

Alternatively, $(D - 2I)e^{-2x} = -2e^{-2x} - 2e^{-2x} = -4e^{-2x}$, so that

$$(D - 2I)^2 e^{-2x} = (D - 2I)(-4e^{-2x}) = 8e^{-2x} + 8e^{-2x} = 16e^{-2x}.$$

9. Differential operators, general solution. The optional Sec. 2.3 introduces the operator notation and shows how it can be applied to linear ODEs with constant coefficients. The given ODE is

$$(D^2 - 4.20D + 4.41I)y = (D - 2.10I)^2 y = y'' - 4.20y' + 4.41y = 0.$$

From this we conclude that a general solution is

$$y = (c_1 + c_2 x)e^{2.10x}$$

because

$$(D - 2.10I)^2((c_1 + c_2 x)e^{2.10x}) = 0.$$

We verify this directly as follows:

$$(D - 2.10I)c_1 e^{2.10x} = 2.10c_1 e^{2.10x} - 2.10c_1 e^{2.10x} = 0$$

and

$$(D - 2.10I)c_2xe^{2.10x} = c_2(D - 2.10I)xe^{2.10x}$$
$$= c_2(e^{2.10x} + 2.10xe^{2.10x} - 2.10xe^{2.10x}) = c_2e^{2.10x},$$

so that

$$(D - 2.10I)^2c_2xe^{2.10x} = (D - 2.10I)c_2e^{2.10x} = c_2(2.10e^{2.10x} - 2.10e^{2.10x}) = 0.$$

Sec. 2.4 Modeling of Free Oscillations of a Mass–Spring System

Newton's law and Hooke's law give the model, namely, the ODE (3), on p. 63, if the damping is negligibly small over the time considered, and (5), on p. 64, if there is damping that cannot be neglected so the model must contain the damping term cy'.

It is remarkable that the three cases in Sec. 2.2 here correspond to three cases in terms of mechanics; see p. 65. The curves in Cases I and II look similar, but their formulas (7) and (8) are different.

Case III includes, as a limiting case, harmonic oscillations (p. 63) in which no damping is present and no energy is taken from the system, so that the motion becomes periodic with the same maximum amplitude C in (4*) at all times. Equation (4*) also shows the phase shift δ. Hence it gives a better impression than the sum (4) of sines and cosines.

The justification of (4*), suggested in the text, begins with

$$y(t) = C\cos(\omega_0 t - \delta) = C(\cos\omega_0 t\cos\delta + \sin\omega_0 t\sin\delta)$$
$$= C\cos\delta\cos\omega_0 t + C\sin\delta\sin\omega_0 t = A\cos\omega_0 t + B\sin\omega_0 t.$$

By comparing, we see that $A = C\cos\delta$, $B = C\sin\delta$, hence

$$A^2 + B^2 = C^2\cos^2\delta + C^2\sin^2\delta = C^2$$

and

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{C\sin\delta}{C\cos\delta} = \frac{B}{A}.$$

Problem Set 2.4. Page 69

5. Springs in parallel. This problem deals with undamped motion and follows the method of Example 1. We have $m = 5$ [kg].

(i) The spring constant [in Hooke's law (2)] is $k_1 = 20$ [nt/m]. Thus the desired frequency of vibration is

$$\frac{w_0}{2\pi} = \frac{\sqrt{k_1/m}}{2\pi} = \frac{\sqrt{20/5}}{2\pi}\ [\text{Hz}] = \frac{1}{\pi}\ [\text{Hz}] = 0.3183\ [\text{Hz}].$$

(ii) Here the frequency of vibration is $\sqrt{45/5}/(2\pi)$ [Hz] $= 3/2\pi$ [Hz] $= 0.4775$ [Hz].

(iii) Let K denote the modulus of the springs in parallel. Let F be some force that stretches the combination of springs by an amount s_0. Then

$$F = Ks_0.$$

Let $k_1 s_0 = F_1$, $k_2 s_0 = F_2$. Then

$$F = F_1 + F_2 = (k_1 + k_2)s_0.$$

By comparison,

$$K = k_1 + k_2 = 20 + 45 = 65 \text{ [nt/m]}$$

so that the desired frequency of vibrations is

$$\frac{\sqrt{K/m}}{2\pi} = \frac{\sqrt{65/5}}{2\pi} = \frac{\sqrt{13}}{2\pi} = 0.5738 \text{ [Hz]}.$$

7. Pendulum. In the solution given on p. A7, the second derivative θ'' is the angular acceleration, hence $L\theta''$ is the acceleration and $mL\theta''$ is the corresponding force. The restoring force is caused by the force of gravity $-mg$ whose tangential component $-mg \sin\theta$ is the restoring force and whose normal component $-mg\cos\theta$ has the direction of the rod in Fig. 42, p. 63. Also $\omega_0^2 = g/L$ is the analog of $\omega_0^2 = k/m$ in (4) because the models are

$$\theta'' + \frac{g}{L}\theta = 0 \qquad \text{and} \qquad y'' + \frac{k}{m}y = 0.$$

13. Initial value problem. The general formula follows from

$$y = (c_1 + c_2 t)e^{-\alpha t}, \qquad y' = [c_2 - \alpha(c_1 + c_2 t)]e^{-\alpha t}$$

by straightforward calculation, solving one equation after the other. First, $y(0) = c_1 = y_0$ and then

$$y'(0) = c_2 - \alpha c_1 = c_2 - \alpha y_0 = v_0, \qquad c_2 = v_0 + \alpha y_0.$$

Together we obtain the answer given on p. A7.

15. Frequency. The binomial theorem with exponent $\frac{1}{2}$ gives

$$\begin{aligned}(1+a)^{1/2} &= \binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{1}a + \binom{\frac{1}{2}}{2}a^2 + \cdots \\ &= 1 + \frac{1}{2}a + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}a^2 + \cdots .\end{aligned}$$

Applied in (9), it gives

$$\omega^* = \left(\frac{k}{m} - \frac{c^2}{4m^2}\right)^{1/2} = \left(\frac{k}{m}\right)^{1/2}\left(1 - \frac{c^2}{4mk}\right)^{1/2} \approx \left(\frac{k}{m}\right)^{1/2}\left(1 - \frac{c^2}{8mk}\right).$$

For Example 2, III, it gives $\omega^* = 3(1 - 100/(8 \cdot 10 \cdot 90)) = 2.9583$ (exact 2.95833).

Sec. 2.5 Euler–Cauchy Equations

This is another large class of ODEs that can be solved by algebra, leading to single powers and logarithms, whereas for constant-coefficient ODEs we obtained exponential and trigonometric functions.

Three cases appear, as for those other ODEs, and Fig. 48, p. 73, gives an idea of what kind of solution we can expect. In some cases $x = 0$ must be excluded (when we have a power with a negative exponent), and in other cases the solutions are restricted to positive values for the independent variable x; this happens when a logarithm or a root appears (see Example 1, p. 71). Note further that the auxiliary equation for determining exponents m in $y = x^m$ is

$$m(m-1) + a\,m + b = 0, \quad \text{thus } m^2 + (a-1)\,m + b = 0,$$

with $a - 1$ as the coefficient of the linear term. Here the ODE is written

$$x^2 y'' + axy' + by = 0, \tag{1}$$

which is no longer in the standard form with y'' as the first term.

Whereas constant-coefficient ODEs are basic in mechanics and electricity, Euler–Cauchy equations are less important. A typical application is shown on p. 73.

In summary, we can say that the key approach to solving the Euler–Cauchy equation is the auxiliary equation $m(m-1) + am + b = 0$. From this most of the material develops.

Problem Set 2.5. Page 73

3. General solution. Double root (Case II). Problems 2–11 are solved, as explained in the text, by determining the roots of the auxiliary equation (3). The ODE $5x^2y'' + 23xy' + 16.2y = 0$ has the auxiliary equation

$$5m(m-1) + 23m + 16.2 = 5m^2 + 18m + 16.2 = 5[(m+1.8)(m+1.8)] = 0.$$

According to (6), p. 72, a general solution for positive x is

$$y = (c_1 + c_2 \ln x)x^{-1.8}.$$

5. Complex roots. The ODE $4x^2y'' + 5y = 0$ has the auxiliary equation

$$4m(m-1) + 5 = 4m^2 - 4m + 5 = 4\big(m - \big(\tfrac{1}{2} + i\big)\big)\big(m - \big(\tfrac{1}{2} - i\big)\big) = 0.$$

A basis of complex solutions is $x^{(1/2)+i}$, $x^{(1/2)-i}$. From it we obtain real solutions by a trick that introduces exponential functions, namely, by first writing (Euler's formula!)

$$x^{(1/2)+i} = x^{1/2}x^{\pm i} = x^{1/2}e^{\pm i \ln x} = x^{1/2}(\cos(\ln x) \pm i \sin(\ln x))$$

and then taking linear combinations to obtain a real basis of solutions

$$\sqrt{x}\cos(\ln x) \qquad \text{and} \qquad \sqrt{x}\sin(\ln x)$$

for positive x or writing $\ln |x|$ if we want to admit all $x \neq 0$.

7. Real roots. The ODE is in D-notation, with D the differential operator from Sec. 2.3. In regular notation we have

$$(x^2D^2 - 4xD + 6I)y = x^2D^2y - 4xDy - 6Iy = x^2y'' - 4xy' + 6y = 0.$$

Using the method of Example 1 of the text and determing the roots of the auxiliary equation (3) we obtain

$$m(m-1) - 4m + 6 = m^2 - 5m + 6 = (m-2)(m-3) = 0$$

and from this the general solution $y = c_1x^2 + c_2x^3$ valid for all x follows.

15. Initial value problem. Initial values cannot be prescribed at $x = 0$ because the coefficients of an Euler–Cauchy equation in standard form $y'' + (a/x)y' + (b/x^2)y = 0$ are infinite at $x = 0$. Choosing $x = 1$ makes the problem simpler than other values would do because $\ln 1 = 0$. The given ODE

$$x^2y'' + 3xy' + y = 0$$

has the auxiliary equation

$$m(m-1) + 3m + 1 = m^2 + 2m + 1 = (m+1)(m+1) = 0$$

which has a double root -1 as a solution. A general solution for positive x, involving the corresponding real solutions, is

$$y = (c_1 + c_2 \ln x)x^{-1}.$$

We need to compute the derivative

$$y' = c_2x^{-2} - c_1x^{-2} - c_2x^{-2} \ln x.$$

Inserting the second initial condition of $y'(1) = 0.4$ into our freshly computed derivative gives

$$y'(1) = c_2 - c_1 = 0.4, \text{ so that } c_2 = 0.4 + c_1.$$

Similarly, the first initial condition $y(1) = 3.6$ is subsituted into the general solution (recall that $\ln 1 = 0$) which gives

$$y(1) = (c_1 + c_2 \ln 1) \cdot 1 = c_1 = 3.6.$$

Plugging this back into the equation for c_2 gives $c_2 = 0.4 + 3.6 = 4$ and hence the solution to the IVP, that is,

$$y = (3.6 + 4 \ln x)x^{-1}.$$

Sec. 2.6 Existence and Uniqueness of Solutions. Wronskian

This section serves as a preparation for the study of higher order ODEs in Chap. 3. You have to understand the Wronskian and its use. The **Wronskian** $W(y_1, y_2)$ of two solutions y_1 and y_2 of an ODE is defined by (6), p. 75. It is conveniently written as a second-order determinant (but this is not essential for using it; you need not be familiar with determinants here). It serves for checking linear independence or dependence, which is important in obtaining bases of solutions. The latter are needed, for instance, in connection with initial value problems, where a single solution will generally not be sufficient for satisfying two given initial conditions. Of course, two functions are linearly independent if and only if their quotient is not constant. To check this, you would not need Wronskians, but we discuss them here in the simple case of second-order ODEs as a preparation for Chapter 3 on higher order ODEs, where Wronskians will show their power and will be very useful.

You should memorize the formula for the determinant of a 2×2 matrix with any entries a, b, c, d given below and the determinant of A, denoted by $\det A$ is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \det\ A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Problem Set 2.6. Page 79

1. Simpler formulas for calculating the Wronskian. Derivation of (6*)(a), (6*)(b) on p. 76 from (6) on p. 75. By the quotient rule of differentiation we have

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_2' y_1 - y_2 y_1'}{y_1^2}.$$

Now

$$W(y_1,\ y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1',$$

but this is precisely the numerator of the first equation so that

$$\left(\frac{y_2}{y_1}\right)' = \frac{W(y_1,\ y_2)}{y_1^2}.$$

Multiplying both sides by y_1^2 gives formula (6*) (a) on p. 76.

To prove formula (6*) (b) we differentiate the negative reciprocal and absorb the minus sign into the numerator to get the Wronskian

$$-\left(\frac{y_1}{y_2}\right)' = -\frac{y_1' y_2 - y_1 y_2'}{y_1^2} = \frac{W(y_1,\ y_2)}{y_1^2}.$$

Solving for $W(y_1,\ y_2)$ gives us formula (6*) (b) immediately.

3. Homogeneous ODE with constant coefficients. There are three ways to compute the Wronskian.

a. *By determinants*. Directly from the definition of the Wronskian, and using the chain rule in differentiation we get

$$W\left(e^{-0.4x},\ e^{-2.6x}\right) = \begin{vmatrix} e^{-0.4x} & e^{-2.6x} \\ (e^{-0.4x})' & (e^{-2.6x})' \end{vmatrix} = \begin{vmatrix} e^{-0.4x} & e^{-2.6x} \\ -0.4e^{-0.4x} & -2.6e^{-2.6x} \end{vmatrix}$$
$$= (e^{-0.4x})(-2.6e^{-2.6x}) - (e^{-2.6x})(-0.4e^{-0.4x}) = -2.2e^{-3x}.$$

b. *By (6*)(a), p.* 76. *Ratios.* We need the ratio of y_2 to y_1 and the derivative of that fraction to compute the Wronskian.

$$\frac{y_2}{y_1} = \frac{e^{-2.6x}}{e^{-0.4x}} = e^{-2.6x-(-0.4x)} = e^{-2.2x}, \qquad \left(\frac{y_2}{y_1}\right)' = -2.2e^{-2.2x}.$$
$$W(y_1, y_2) = \left(\frac{y_2}{y_1}\right)' y_1^2 = (-2.2e^{-2.2x})(e^{-0.4x})^2 = -2.2e^{-3x}.$$

c. *By (6*)(b), p.* 76. *Ratios.* The second formula is computed similarly, that is,

$$\frac{y_1}{y_2} = \frac{e^{-0.4x}}{e^{-2.6x}} = e^{2.2x}, \qquad -\left(\frac{y_1}{y_2}\right)' = -2.2e^{2.2x},$$
$$W(y_1, y_2) = -\left(\frac{y_1}{y_2}\right)' y_1^2 = -2.2e^{2.2x}(e^{-0.4x})^2 = -2.2e^{-3x}.$$

Note that the two solutions $y_1(x) = e^{-0.4x}$, $y_2(x) = e^{-2.6x}$ are solutions to a second-order, homogeneous, linear ODE (1) with constant coefficients of Sec. 2.2. We can compare to (6) in Sec. 2.2 to see that the general solution is

$$y = c_1 e^{\lambda_1} + c_2 e^{\lambda_2} = c_1 e^{-0.4x} + c_2 e^{-2.6x}$$

so that we have two distinct roots $\lambda_1 = -0.4$ and $\lambda_2 = -2.6$. The corresponding characteristic equation is

$$(\lambda + 0.4)(\lambda + 2.6) = \lambda^2 + 3\lambda + 1.04 = 0,$$

and the corresponding ODE is

$$y'' + 3y' + 1.04y = 0.$$

The quotient $y_1/y_2 = e^{2.2x}$ is not a constant so y_1 and y_2 form a basis for a solution as is duly noted on the bottom of p. 54 (in the first paragraph of Case I of Sec. 2.2). Hence y_1 and y_2 are linearly independent by definition of a basis. (Note we actually gave the ODE whose basis is $y_1(x) = e^{-0.4x}$, $y_2(x) = e^{-2.6x}$). Furthermore, by Theorem 2 of Sec. 2.6 $W = -2.2e^{-3x} \neq 0$ for all x, so that we have linear independence.

5. Euler–Cauchy equation. Computing the Wronskian by determinants we get

$$W(x^3,\ x^2) = \begin{vmatrix} x^3 & x^2 \\ (x^3)' & (x^2)' \end{vmatrix} = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix}$$
$$= x^3 2x - x^2 3x^2 = -x^4.$$

corresponding to the answer in Appendix A on p. A7. The second approach by (6*)(a) is

$$W = \left(\frac{y_2}{y_1}\right)' y_1^2 = \left(\frac{x^2}{x^3}\right)' (x^3)^2 = \left(\frac{1}{x}\right)' (x^6) = (-x^{-2})(x^6) = -x^4.$$

Similarly for (6*)(b). The two solutions x^3 and x^2 belong to an ODE of the Euler–Cauchy type of Sec. 2.5. By (4), Sec. 2.5, the roots of the characteristic equation are 3 and 2. Using (2), Sec. 2.5, its characteristic equation is

$$(m-3)(m-2) = m^2 - 5m + 6 = m(m-1) - 4m + 6 = 0.$$

Hence the ODE is $x^2y'' - 4xy' + 6y = 0$. The quotient $y_1/y_2 = x^3/x^2 = x$ is not a constant so by the text before (4) of Sec. 2.5, x^3 and x^2 are linearly independent. Also $W = -x^{-4} \neq 0$, which by Theorem 2 on p. 75 shows linear independence on any interval.

9. Undamped oscillation. (a) By Example 6 of Sec. 2.2, or equation (4) of Sec. 2.4 or Example 1 of Sec. 2.6 (we have 3 ways to see this!) we know that $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$ are solutions of the homogeneous linear ODE with constant coefficients and complex conjugate roots (Case III) $y'' + \omega^2 y = 0$. Here $\omega = 5$, so that the desired ODE is $y'' + 25y = 0$.

(b) To show linear independence compute the Wronskian and get

$$W(\cos 5x,\ \sin 5x) = \begin{vmatrix} \cos 5x & \sin 5x \\ -5\sin 5x & 5\cos 5x \end{vmatrix} = 5(\cos^2 5x + \sin^2 5x) = 5 \cdot 1 = 5 \neq 0$$

so that, by Theorem 2, the given functions are linearly independent.

(c) The general solution to the ODE is $y(x) = A\cos 5x + B\sin 5x$, so that $y(0) = A = 3$. Next we take the derivative $y'(x) = -5A\sin 5x + 5B\sin 5x$, so that $y'(0) = 5B = -5$ and $B = -1$. Hence the solution to the given IVP is $y = 3\cos 5x - \sin 5x$, as in the solution on p. A7.

13. Initial value problem. (a) The first given function is 1 which is equal to e^{0x}. Hence, for the given e^{0x} and e^{-2x}, the corresponding characteristic equation and then the corresponding ODE are

$$(\lambda - 0)(\lambda + 2) = \lambda^2 + 2\lambda = 0, \qquad y'' + 2y' = 0.$$

(b) To show linear independence, we compute the Wronskian

$$W(1,\ e^{-2x}) = \begin{vmatrix} 1 & e^{-2x} \\ 0 & -2e^{-2x} \end{vmatrix} = -2e^{-2x} \neq 0.$$

Hence, by Theorem 2, the functions 1 and e^{-2x} are linearly independent.

(c) The general solution to the ODE is $y(x) = c_1 + c_2 e^{-2x}$, so that $y(0) = c_1 + c_2 = 1$. Next we take the derivative $y'(x) = -2c_2e^{-2x}$, so that $y'(0) = -2c_2 = -1$. Hence $c_2 = \frac{1}{2}$. This gives $c_1 = 1 - c_2 = 1 - \frac{1}{2} = \frac{1}{2}$. Hence the solution to the IVP is

$$y = \tfrac{1}{2} + \tfrac{1}{2}e^{-2x} = 0.5(1 + e^{-2x}).$$

Sec. 2.7 Nonhomogeneous ODEs

This section and problem set deal with nonhomogeneous linear ODEs

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

where $r(x)$ is not identically zero $[r(x) \not\equiv 0]$. The new task is the determination of a particular solution y of (1). For this we can use the method of undetermined coefficients. Because of the Modification Rule, it is necessary to *first* determine a general solution of the homogeneous ODE since the form of y_p differs depending on whether or not the function (or a term of it) on the right side of the ODE is a solution of the homogeneous ODE. If we forget to take this into account, we will not be able to determine the coefficients; in this sense the method will warn us that we made a mistake.

Problem Set 2.7. Page 84

1. General solution. The characteristic equation of the homogeneous ODE $y'' + 5y' + 4y = 0$ is

$$\lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4) = 0.$$

We see that it has the solutions -1 and -4. Hence a general solution of the homogeneous ODE $y'' + 5y' + 4y = 0$ is

$$y_h = c_1e^{-x} + c_2e^{-4x}.$$

The function $10\,e^{-3x}$ on the right is not a solution of the homogeneous ODE. Hence we do not apply the Modification Rule. Table 2.1, p. 82, requires that we start from $y_p = Ce^{-3x}$. Two differentiations give

$$y_p' = -3Ce^{2x} \qquad \text{and} \qquad y_p'' = 9Ce^{-3x}.$$

Substitution of y_p and its derivatives into the given nonhomogeneous ODE yields

$$9Ce^{-3x} + 5 \cdot (-3Ce^{-3x}) + 4Ce^{-3x} = 10e^{-3x}.$$

Simplifying gives us

$$e^{-3x}(9C - 15C + 4C) = 10e^{-3x}.$$

Hence $-2C = 10$, $C = 10/(-2) = -5$. This gives the answer (a general solution of the given ODE; see page A7)

$$y = c_1 e^{-x} + c_2 e^{-4x} - 5e^{-3x}.$$

9. Modification rule. Additive split of particular solution. Recalling that D stands for differential operator (review Sec. 2.3), we can write the given nonhomogeneous ODE as

$$y'' - 16y = 9.6e^{4x} + 30e^{x}.$$

The homogeneous ODE is $y'' - 16y = 0$ and its characteristic equation is $\lambda^2 - 16 = (\lambda - 4) \cdot (\lambda + 4) = 0$. Its roots are 4 and -4. Hence a real general solution of the homogeneous ODE is

$$y_h = c_1 e^{4x} + c_2 e^{-4x}.$$

On the right-hand side of the given nonhomogeneous ODE we have

$$9.6e^{4x} + 30e^{x}$$

and see that $9.6e^{4x}$ is a solution of the homogeneous ODE. Hence the modification rule (b) applies to that term. We split additively

$$y_p = y_{p_1} + y_{p_2}$$

where by Table 2.1 on p. 82 we determine y_{p_1} and y_{p_2} and obtain

$$y_{p_1} = C_1 xe^{4x}, \qquad y_{p_2} = C_2 e^{x},$$

(the factor x in y_{p_1} is a direct consequence of the modification rule).

By differentiation of y_{p_1}, using the chain rule,

$$\begin{aligned} y'_{p_1} &= C_1 e^{4x} + 4C_1 xe^{4x} \\ y''_{p_1} &= 4C_1 e^{4x} + 4C_1 e^{4x} + 16C_1 xe^{4x} = 8C_1 e^{4x} + 16C_1 xe^{4x}. \end{aligned}$$

We do the same for y_{p_2}

$$y'_{p_2} = C_2 e^{x}, \qquad y''_{p_2} = C_2 e^{x}.$$

We now substitute y''_{p_1} and y_{p_1} into the ODE $y'' - 16\,y = 9.6e^{4x}$ and get

$$8C_1 e^{4x} + 16C_1 xe^{4x} - 16C_1 xe^{4x} = 9.6e^{4x}.$$

Hence $8C_1e^{4x} = 9.6e^{4x}$, so that $C_1 = 9.6/8 = 1.2$. Similarly, substitute y''_{p_2} and y_{p_2} into the ODE $y'' - 16y = 30e^x$ and get

$$C_2e^x - 16C_2e^x = 30e^x.$$

Hence $-15C_2e^x = 30e^x$, so that $C_2 = -\frac{30}{15} = -2$. The desired general solution is

$$y = c_1e^{4x} + c_2e^{-4x} + 1.2xe^{4x} - 2e^x.$$

11. Initial value problem. The homogeneous ODE $y'' + 3y = 0$ has the characteristic equation $\lambda^2 + 3 = 0$. From this, we immediately see that its roots are $\lambda = -i\sqrt{3}$ and $i\sqrt{3}$. Hence a general solution of the homogeneous ODE (using Case III of table "Summarizing Cases I–III" in Sec. 2.2) gives

$$y_h = A\cos\sqrt{3}x + B\sin\sqrt{3}x.$$

Since the right-hand side of the given nonhomogeneous ODE is not a solution of the homogeneous ODE, the modification rule does not apply. By Table 2.1 (second row with $n = 2$) we have

$$y_p = K_2x^2 + K_1x + K_0.$$

Differentiating twice gives us

$$y'_p = 2K_2x + K_1, \qquad y''_p = 2K_2.$$

Substitute y''_p and y_p into the given nonhomogeneous ODE (which has no y' term and so y'_p has no place to be substituted) and group by exponents of x

$$2K_2 + 3(K_2x^2 + K_1x + K_0) = 18x^2.$$
$$3K_2x^2 + 3K_1x + 2K_2 + 3K_0 = 18x^2.$$

Now compare powers of x on both sides. The x^2-terms give $3K_2 = 18$, so that $K_2 = 6$. Furthermore, the x-terms give $3K_1 = 0$ since there is no x-term on the right. Finally the constant terms give (substituting $K_2 = 6$)

$$2K_2 + 3K_0 = 0,\ 2\cdot 6 + 3K_0 = 0 \qquad \text{and} \qquad K_0 = -\frac{12}{3} = -4.$$

Hence the general solution of the given nonhomogeneous ODE is

$$y = y_h + y_p = A\cos\sqrt{3}x + B\sin\sqrt{3}x + 6x^2 - 4.$$

Only now can we consider the initial conditions. (Why not earlier?) The first condition is $y(0) = -3$ and gives

$$y(0) = A\cdot 1 + B\cdot 0 + 0 - 4 = -3,\ \text{hence } A = 1.$$

For the second initial condition $y'(0) = 0$ we need the derivative of the general solution (using the chain rule)

$$y' = -A\sqrt{3}\sin\sqrt{3}x + B\sqrt{3}\cos\sqrt{3}x + 12x$$

and its value

$$y'(0) = 0 + B\sqrt{3} \cdot 1 + 0 = 0 \text{ hence } B = 0.$$

Putting $A = 1$ and $B = 0$ into the general solution gives us the solution to the IVP

$$y = \cos\sqrt{3}x + 6x^2 - 4.$$

Sec. 2.8 Modeling: Forced Oscillations. Resonance

In the solution a, b of (4) in the formula on p. 87 before (5) (the formula without a number) the denominator is the coefficient determinant; furthermore, for a the numerator is the determinant

$$\begin{vmatrix} F_0 & \omega c \\ 0 & k - m\omega^2 \end{vmatrix} = F_0(k - m\omega^2).$$

Similarly for b, by Cramer's rule or by elimination.

Problem Set 2.8. Page 91

3. Steady-state solution. Because of the function $r(t) = 42.5\cos 2t$, we have to choose

$$y_p = K\cos 2t + M\sin 2t.$$

By differentiation,

$$\begin{aligned} y_p' &= -2K\sin 2t + 2M\cos 2t \\ y_p'' &= -4K\cos 4t - 4M\sin 4t. \end{aligned}$$

Substitute this into the ODE $y'' + 6y' + 8y = 42.5\cos 2t$. To get a simple formula, use the abbreviations $C = \cos 2t$ and $S = \sin 2t$. Then

$$(-4KC - 4MS) + 6(-2KS + 2MC) + 8(KC + MS) = 42.5C.$$

Collect the C-terms and equate their sum to 42.5. Collect the S-terms and equate their sum to 0 (because there is no sine term on the right side of the ODE). We obtain

$$\begin{aligned} -4K + 12M + 8K &= 4K + 12M = 42.5 \\ -4M - 12K + 8M &= -12K + 4M = 0. \end{aligned}$$

From the second equation, $M = 3K$. Then from the first equation,

$$4K + 12 \cdot 3K = 42.5, \qquad K = 42.5/40 = 1.0625.$$

Hence $M = 3K = 3 \cdot 1.0625 = 3.1875$. The steady-state solution is (cf. p. A8)

$$y = 1.0625\cos 2t + 3.1875\sin 2t.$$

9. Transient solution. The homogeneous ODE $y'' + 3y' + 3.25y = 0$ has the characteristic equation

$$\lambda^2 + 3\lambda + 3.25 = (\lambda + 1.5 - i)(\lambda + 1.5 + i) = 0.$$

Hence the roots are complex conjugates: $-1.5 + i$ and $-1.5 - i$. A general solution of the homogeneous ODE (see Sec. 2.2., table on p. 58, Case III, with $\omega = 1$, $a = -3$) is

$$y_h = e^{-1.5t}(A \cos t + B \sin t).$$

To obtain a general solution of the given ODE we need a particular solution y_p. According to the method of undetermined coefficients (Sec. 2.7) set

$$y_p = K \cos t + M \sin t.$$

Differentiate to get

$$y_p' = -K \sin t + M \cos t$$
$$y_p'' = -K \cos t - M \sin t.$$

Substitute this into the given ODE. Abbreviate $C = \cos t$, $S = \sin t$. We obtain

$$(-KC - MS) + 3(-KS + MC) + 3.25(KC + MS) = 3C - 1.5S.$$

Collect the C-terms and equate their sum to 3

$$-K + 3M + 3.25K = 3 \quad \text{so that } 2.25K + 3M = 3.$$

Collect the S-terms and equate their sum to -1.5:

$$-M - 3K + 3.25M = -15 \quad \text{so that } -3K + 2.25M = -1.5.$$

We solve the equation for the C-terms for M and get $M = 1 - 0.75K$. We substitute this into the previous equation (equation for the S-terms), simplify, and get a value for K

$$2.25(1 - 0.75K) - 3K = -1.5 \quad \text{thus } K = 0.8.$$

Hence

$$M = 1 - 0.75K = 1 - 0.75 \cdot 0.8 = 0.4.$$

This confirms the answer on p. A8 that the transient solution is

$$y = y_h + y_p = e^{-1.5t}(A \cos t + B \sin t) + 0.8 \cos t + 0.4 \sin t.$$

17. Initial value problem. The homogeneous ODE is $y'' + 4y = 0$. Its characteristic equation is

$$\lambda^2 + 4 = (\lambda - 2)(\lambda + 2) = 0.$$

It has the roots 2 and -2, so that a general solution of the homogeneous ODE is

$$y_h = c_1 e^{2t} + c_2 e^{-2t}.$$

Next we need a particular solution y_p of the given ODE. The right-hand side of that ODE is $\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \cos 5t$. Using the method of undetermined coefficients (Sec. 2.7), set $y_p = y_{p_1} + y_{p_2} + y_{p_3}$.

For the first part of y_p

$$y_{p_1} = K_1 \sin t + M_1 \cos t.$$

Differentiation (chain rule) then gives

$$y'_{p_1} = K_1 \cos t - M_1 \sin t, \qquad y''_{p_1} = -K_1 \sin t - M_1 \cos t.$$

Similarily for y_{p_2}:

$$y_{p_2} = K_2 \sin 3t + M_2 \cos 3t.$$
$$y'_{p_2} = 3K_2 \cos 3t - 3M_2 \sin 3t, \qquad y''_{p_2} = -9K_2 \sin 3t - 9M_2 \cos 3t.$$

Finally, for y_{p_3}, we have

$$y_{p_3} = K_3 \sin 5t + M_3 \cos 5t.$$
$$y'_{p_3} = 5K_3 \cos 5t - 5M_3 \sin 5t, \qquad y''_{p_3} = -25K_3 \sin 5t - 25M_3 \cos 5t.$$

Denote $S = \sin\, t$, $C = \cos\, t$; $S^* = \sin 3t$, $C^* = \cos 3t$; $S^{**} = \sin 5t$, $C^{**} = \cos 5t$. Substitute y_p and $y''_p = y''_{p_1} + y''_{p_2} + y''_{p_3}$ (with the notation S, C, S^*, C^*, S^{**}, and C^{**}) into the given ODE

$$y'' + 4y = \sin t + \tfrac{1}{3} \sin 3t + \tfrac{1}{5} \cos 5t$$

and get a very long equation, stretching over two lines:

$$\begin{aligned}&(-K_1S - M_1C - 9K_2S^* - 9M_2C^* - 25K_3S^{**} - 25M_3C^{**})\\ &\quad + (4K_1S + 4M_1C + 4K_2S^* + 4M_2C^* + 4K_3S^{**} + 4M_3C^{**}) = S + \tfrac{1}{3}S^* + \tfrac{1}{5}C^{**}.\end{aligned}$$

Collect the S-terms and equate the sum of their coefficients to 1 because the right-hand side of the ODE has one term $\sin t$, which we denoted by S.

$$[S\text{-terms}] \qquad -K_1 + 4K_1 = 1 \qquad \text{so that } K_1 = \tfrac{1}{3}.$$

Simlarly for C-terms

$$[C\text{-terms}] \qquad -M_1 + 4M_1 = 0 \qquad \text{so that } M_1 = 0.$$

Then for S^*-terms, C^*-terms

$$[S^*\text{-terms}] \qquad -9K_2 + 4K_2 = \tfrac{1}{3} \qquad \text{so that } K_2 = -\tfrac{1}{15}$$
$$[C^*\text{-terms}] \qquad -9M_2 + 4M_2 = 0 \qquad \text{so that } M_2 = 0.$$

And finally for S^{**}-terms, C^{**}-terms

$$[S^{**}\text{-terms}] \qquad -25K_3 + 4K_3 = \tfrac{1}{5} \qquad \text{so that } K_3 = -\tfrac{1}{105}$$
$$[C^{**}\text{-terms}] \qquad -25M_3 + 4M_3 = 0 \qquad \text{so that } M_3 = 0.$$

Hence the general solution of the given ODE is

$$y(t) = c_1e^{2t} + c_2e^{-2t} + \tfrac{1}{3} \sin t - \tfrac{1}{15} \sin 3t - \tfrac{1}{105} \sin 5t.$$

We now consider the initial conditions. The first condition $y(0) = 0$ gives $y(0) = c_1 + c_2 + 0 = 0$, so that $\boxed{c_1 = -c_2}$. For the second condition $y'(0) = \frac{3}{35}$, we need to compute $y'(t)$

$$y'(t) = 2c_1 e^{2x} - 2c_2 e^{-2x} + \tfrac{1}{3}\cos t - \tfrac{3}{15}\cos 3t - \tfrac{5}{105}\cos 5t.$$

Thus we get (noting that the fractions on the left-hand side add up to $\frac{1}{3} - \frac{3}{15} - \frac{5}{105} = \frac{5\cdot 7 - 3\cdot 7 - 5}{3\cdot 5\cdot 7} = \frac{9}{3\cdot 5\cdot 7} = \frac{3}{5\cdot 7} = \frac{3}{35}$)

$$y'(0) = 2c_1 - 2c_2 + \tfrac{1}{3} - \tfrac{3}{15} - \tfrac{5}{105} = \tfrac{3}{35}, \qquad y'(0) = 2c_1 - 2c_2 = 0, \qquad \boxed{c_1 = c_2}.$$

The only way the two "boxed" equations for $c_1$and c_2 can hold *at the same time* is for $c_1 = c_2 = 0$. Hence we get the answer to the IVP

$$y(t) = \tfrac{1}{3}\sin t - \tfrac{1}{15}\sin 3t - \tfrac{1}{105}\sin 5t.$$

Sec. 2.9 Modeling: Electric Circuits

Problem Set 2.9. Page 98

5. ***LC*-circuit.** Modeling of the circuit is the same as for the *RLC*-circuit. Thus,

$$LI' + Q/C = E(t).$$

By differentiation,

$$LI'' + I/C = E'(t).$$

Here $L = 0.5$, $1/C = 200$, $E(t) = \sin t$, $E'(t) = \cos t$, so that

$$0.5I'' + 200I = \cos t, \quad \text{thus } I'' + 400I = 2\cos t.$$

The characteristic equation $\lambda^2 + 400 = 0$ has the roots $\pm 20i$, so that a real solution of the homogeneous ODE is

$$I_h = c_1 \cos 20t + c_2 \sin 20t.$$

Now determine a particular solution I_p of the nonhomogeneous ODE by the method of undetermined coefficients in Sec. 2.7, starting from

$$I_p = K\cos t + M\sin t.$$

Substitute this and the derivatives

$$I_p' = -K\sin t + M\cos t$$
$$I_p'' = -K\cos t - M\sin t$$

into the nonhomogeneous ODE $I'' + 400I = 2\cos t$, obtaining

$$(-K\cos t - M\sin t) + 400(K\cos t + M\sin t) = 2\cos t.$$

Equating the cosine terms on both sides, we get

$$-K + 400K = 2, \quad \text{hence } K = \tfrac{2}{399}.$$

The sine terms give

$$-M + 400M = 0, \quad \text{hence } M = 0.$$

We thus have the general solution of the nonhomogeneous ODE

$$I = I_h + I_p = c_1 \cos 20t + c_2 \sin 20t + \tfrac{2}{399} \cos t.$$

Now use the initial conditions $I(0) = 0$ and $I'(0) = 0$, to obtain

$$I(0) = c_1 + \tfrac{2}{399} = 0, \quad \text{thus } c_1 = -\tfrac{2}{399}.$$

Differentiation gives

$$I'(t) = -20c_1 \sin 20t + 20c_2 \cos 20t - \tfrac{2}{399} \sin t.$$

Since $\sin\ 0 = 0$, this gives $I'(0) = 20c_2 = 0$, so that $c_2 = 0$. We thus obtain the answer (cf. p. A8)

$$I(t) = \tfrac{2}{399}(\cos t - \cos 20t).$$

9. Steady-state current. We must find a general solution of the nonhomogeneous ODE. Since $R = 4$, $L = 0.1$, $1/C = 20$, $E = 110$, $E' = 0$, this ODE is

$$0.2I'' + 4I' + 20I = E'(t) = 0.$$

Multiply by 5, to get the standard form with I'' as the first term,

$$I'' + 20I' + 100I = 0.$$

The characteristic equation

$$\lambda^2 + 20\lambda + 100 = (\lambda + 10)^2 = 0$$

has a double root $\lambda = -10$. Hence, by the table of Sec. 2.2 (Case II, real double root), we have the solution to our homogeneous ODE

$$I = (c_1 + c_2 t)e^{-10t} = c_1 e^{-10t} + c_2 t e^{-10t}.$$

As t increases, I will go to 0, since e^{-10t} goes to 0 for t increasing, and $e^{10t} \gg t$, so that $te^{-10t} = t/e^{10t}$ also goes to 0 with e^{-10t} determining the speed of decline (cf. also p. 95). This also means that the steady-state current is 0. Compare this to Problem 11.

11. Steady-state current. We must find a general solution of the nonhomogeneous ODE. Since $R = 12$, $L = 0.4$, $1/C = 80$, $E = 220 \sin 10t$, $E' = 2200 \cos 10t$, this ODE is

$$0.4I'' + 12I' + 80I = E'(t) = 2200 \cos 10t.$$

Multiply by 2.5 to get the standard form with I'' as the first term,

$$I'' + 30I' + 200I = 5500 \cos 10t.$$

The characteristic equation

$$\lambda^2 + 30\lambda + 200 = (\lambda + 20)(\lambda + 10) = 0$$

has roots -20, -10 so that a general solution of the homogeneous ODE is

$$I_h = c_1 e^{-20t} + c_2 e^{-10t}.$$

This will go to zero as t increases, regardless of initial conditions (which are not given in this problem). We also need a particular solution I_p of the nonhomogeneous ODE; this will be the steady-state solution. We obtain it by the method of undetermined coefficients, starting from

$$I_p = K \cos 10t + M \sin 10t.$$

By differentiation we have

$$I_p' = -10K \sin 10t + 10M \cos 10t$$
$$I_p'' = -100K \cos 10t - 100M \sin 10t.$$

Substitute all this into the nonhomogeneous ODE in standard form, abbreviating $C = \cos 10t$, $S = \sin 10t$. We obtain

$$(-100KC - 100MS) + 30(-10KS + 10MC) + 200(KC + MS) = 5500C.$$

Equate the sum of the S-terms to zero,

$$-100M - 300K + 200M = 0, \qquad 100M = 300K, \qquad M = 3K.$$

Equate the sum of the C-terms to 5500 (the right side)

$$-100K + 300M + 200K = 5500, \qquad 100K + 300 \cdot 3K = 1000K = 5500.$$

We see that $K = 5.5$, $M = 3K = 16.5$, and we get the transient current.

$$I = I_h + I_p = c_1 e^{-20t} + c_2 e^{-10t} + 5.5 \cos 10t + 16.5 \sin 10t.$$

Since the transient current $I = I_h + I_p$ tends to the steady-state current I_p (see p. 95) and since this problem wants us to model the steady-state currrent, the final answer (cf. p. A8) is, in [amperes],

$$\text{steady-state current } I_p = 5.5 \cos 10t + 16.5 \sin 10t.$$

Sec. 2.10 Solution by Variation of Parameters

This method is a general method for solving *all* nonhomogeneous linear ODEs. Hence it can solve more problems (e.g., Problem 1 below) than the method of undetermined coefficients in Sec. 2.7, which *is restricted to constant-coefficent ODEs with special right sides.* The method of Sec. 2.10 reduces the problem of solving a linear ODE to that of the evaluation of two integrals and is an extension of the solution method in Sec. 1.5 for first-order ODEs. So why bother with the method of Sec. 2.7? The reason is that the method of undetermined coefficients of Sec. 2.7 is more important to the engineer and physicists than that of Sec. 2.10 because it takes care of cases of the usual periodic driving forces (electromotive forces). Furthermore, the method of Sec. 2.7 is more straightforward to use because it involves only differentiation. The integrals in the method of Sec. 2.10 may be difficult to solve. (This is, of course, irrelvant, if we use a CAS. However, remember, to understand engineering mathematics, we still have to do some exercises by hand!)

Problem Set 2.10. Page 102

1. General solution. Method of solution by variation of parameters needed. Since the right-hand side of the given ODE $y'' + 9y = \sec 3x$ is $\sec 3x$, the method of Sec. 2.7 does not help. (Look at the first column of Table 2.1 on p. 82—the sec function *cannot be found!*) First, we solve the homogeneous ODE $y'' - 9y = 0$ by the method of Sec. 2.1, with the characteristic equation being $\lambda^2 + 9 = 0$, roots $\lambda = \pm 3i$, so that the basis for the solution of the homogeneous ODE is $y_1 = \cos 3x$, $y_2 = \sin 3x$. Hence the general solution for the homogeneous ODE is

$$y_h = \widetilde{A}\cos 3x + \widetilde{B}\sin 3x.$$

The corresponding Wronskian is (using the chain rule, and recalling that $\cos^2 3x + \sin^2 3x = 1$)

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3(\cos^2 3x + \sin^2 3x) = 3.$$

We want to apply (2) on p. 99. We have $y_1 = \cos 3x$, $y_2 = \sin 3x$, $r(x) = \sec 3x = 1/\cos 3x$ (see Appendix A3.1, formula (13)), $W = 3$. We evaluate the two integrals first (where we denoted the arbitrary constant of integration for the first integral by $-c_1$, to have a $+c_1$ in the particular solution)

$$\int \frac{y_2 r}{W}dx = \int \frac{1}{3}\frac{\sin 3x}{\cos 3x}dx = \frac{1}{3}\int \tan 3x\, dx = -\frac{1}{9}\ln|\cos 3x| - c_1,$$

and

$$\int \frac{y_1 r}{W}dx = \int \frac{1}{3}\frac{\cos 3x}{\cos 3x}dx = \frac{1}{3}\int dx = \frac{1}{3}x + c_2.$$

Putting these together for (2)

$$\begin{aligned} y_p(x) &= -y_1\int \frac{y_2 r}{W}dx + y_2\int \frac{y_1 r}{W}dx = (-\cos 3x)\left(-\frac{1}{9}\ln|\cos 3x| - c_1\right) + (\sin 3x)\left(\frac{1}{3}x + c_2\right) \\ &= \frac{1}{9}\cos 3x \ln|\cos 3x| + c_1\cos 3x + \frac{1}{3}x(\sin 3x) + c_2 \sin 3x. \end{aligned}$$

Then the general (final) solution is

$$y = y_h(x) + y_p(x) = A\cos 3x + B\sin 3x + \tfrac{1}{9}\cos 3x \ln|\cos 3x| + \tfrac{1}{3}x(\sin 3x).$$

(Note that, in the final solution, we have absorbed the constants, that is, $A = \widetilde{A} + c_1$ and $B = \widetilde{B} + c_2$. This can always be done, since we can choose the aribitrary constants of integration.)

3. General solution. The solution formula (2) was obtained for the standard form of an ODE. In the present problem, divide the given nonhomogeneous Euler–Cauchy equation

$$x^2y'' - 2xy' + 2y = x^3\sin x$$

by x^2 in order to determine $r(x)$ in (2). We obtain

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x \sin x.$$

The auxiliary equation for the homogeneous ODE is needed to determine y_1 and y_2. It is

$$m(m-1) - 2m + 2 = m^2 - 3m + 2 = (m-2)(m-1) = 0.$$

The roots are 1 and 2. Hence we have the basis $y_1 = x$, $y_2 = x^2$. We also need the corresponding Wronskian

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2.$$

Now use (2) (and integrate by parts in the first integral), obtaining

$$\begin{aligned} y &= -x \int \frac{x^2 \cdot x \sin x}{x^2} dx + x^2 \int \frac{x \cdot x \sin x}{x^2} dx \\ &= -x \int x \sin x \, dx + x^2 \int \sin x \, dx \\ &= -x(\sin x - x \cos x) - x^2 \cos x \\ &= -x \sin x. \end{aligned}$$

Using this particular solution, we obtain the general solution given on p. A8:

$$y = c_1 x + c_2 x^2 - x \sin x.$$

We would obtain this solution directly if we added constants of integrations when we evaluated the integrals.

13. General solution. Choice of method. The given ODE $(x^2D^2 + xD - 9I)y = 48x^5$ is, in the more familiar notation (see Sec. 2.3),

$$x^2y'' + xy' - 9y = 48x^5.$$

We see that the homogeneous ODE $x^2y'' + xy' - 9y = 0$ is of the Euler–Cauchy form (Sec. 2.5). The auxiliary equation is

$$m(m-1) + m - 9 = m^2 - 9 = (m-3)(m+3) = 0,$$

and the solution of the homogeneous ODE is

$$y' = c_1x^{-3} + \widetilde{c}_2x^3.$$

We have a choice of methods. We could try the method of variation of parameters, but it is more difficult than the method of undetermined coefficients. (It shows the importance of the less general method of undetermined coefficients!) From Table 2.1 on p. 82 and the rules on p. 81 we start with

$$y_p = K_5x^5 + K_4x^4 + K_3x^3 + K_2x^2 + K_1x + K_0.$$

We need to differentiate the expression twice and get

$$\begin{aligned} y_p' &= 5K_5x^4 + 4K_4x^3 + 3K_3x^2 + 2K_2x + K_1 \\ y_p'' &= 20K_5x^4 + 12K_4x^2 + 6K_3x + 2K_2. \end{aligned}$$

Subsitute y_p'', y_p', and y_p into the given ODE $x^2y'' + xy' - 9y = 48x^5$ to obtain

$$\begin{aligned} &x^2(20K_5x^4 + 12K_4x^2 + 6K_3x + 2K) + x(5K_5x^4 + 4K_4x^3 + 3K_3x^2 + 2K_2x + K_1) \\ &\quad - 9(K_5x^5 + K_4x^4 + K_3x^3 + K_2x^2 + K_1x + K_0) = 48x^5. \end{aligned}$$

Since there is only the term $48x^5$ on the right-hand side, many of the coefficients will be zero. We have

$$\begin{array}{lrll} [x^0] & -9K_0 = 0 & \text{so that} & K_0 = 0, \\ [x^1] & K_1 - 9K_1 = 0 & \text{so that} & K_1 = 0, \\ [x^2] & 2K_2 + 2K_2 - 9K_2 = 0 & \text{so that} & K_2 = 0, \\ [x^3] & 6K_3 + 3K_3 - 9K_3 = 0 & \text{so that} & K_3 \text{ is arbitary}, \\ [x^4] & 12K_4 + 4K_3 - 9K_4 = 0 & \text{so that} & K_4 = 0, \\ [x^5] & 20K^5 + 5K^5 - 9K^5 = 48 & \text{so that} & K^5 = 3. \end{array}$$

Putting it all together, we get (absorbing $\widetilde{c}_2 + K_3 = c_2$, hence $\widetilde{c}_2x^3 + K_3x^3 = c_2x^3$ for beauty) as our final answer

$$y = y_h + y_p = c_1x^{-3} + c_2x^3.$$

Chap. 3 Higher Order Linear ODEs

This chapter extends the methods for solving homogeneous and nonhomogeneous linear ODEs of second order to methods for solving such ODEs of higher order, that is, *ODEs of order greater than 2*. **Sections 2.1, 2.2, 2.7, and 2.10 as well as Sec. 2.5** (which plays a smaller role than the other four sections) **are generalized.** It is beautiful to see how mathematical methods can be generalized, and it is one hallmark of a good theory when such generalizations can be done. If you have a good understanding of Chap. 2, then solving problems in Chap. 3 will almost come natural to you. However, if you had some problems with Chap. 2, then this chapter will give you further practice with solving ODEs.

Sec. 3.1 Homogeneous Linear ODEs

Section 3.1 is similar to Sec. 2.1. One difference is that we use the Wronskian to show linear dependence or linear independence. Take a look at **Problem 5**, which makes use of the important Theorem 3 on p. 109. The concept of a **basis** is so important because, if you found some functions that are solutions to an ODE *and you can show that they form a basis*—then you know that this solution contains the *smallest* possible number of functions that determine the solution—and from that you can write out the general solution as usual. Also these functions are linearly independent. If you have linear dependence, then at least one of the functions that solves the ODE is superfluous, can be removed, and does not contribute to the general solution. In practice, the methods of solving ODEs when applied carefully would automatically also give you a basis. So usually we do not have to run tests for linear independence and dependence when we apply the methods of Chap. 3 (and Chap. 2) for solving ODEs.

Example 5. Basis, Wronskian. In pulling out the exponential functions, you see that their product equals $e^0 = 1$; indeed,

$$e^{-2x}\, e^{-x}\, e^{x}\, e^{2x} = e^{-2x-x+x+2x} = e^0 = 1.$$

In subtracting columns, as indicated in the text, you get

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -3 & 0 \\ -8 & 7 & 9 & 16 \end{vmatrix}.$$

You can now see the third-order determinant shown in the example. This determinant is simplified as indicated and then developed by the second row:

$$\begin{vmatrix} 1 & 2 & 4 \\ -3 & 0 & 0 \\ 7 & 2 & 16 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 \\ 2 & 16 \end{vmatrix} = 3(32-8) = 72.$$

We discuss more about how to compute higher order determinants in the remark in **Problem 5** of Problem Set 3.1.

Problem Set 3.1. Page 111

1. Basis. A general solution is obtained by four successive integrations:

$$y''' = c_1, \qquad y'' = c_1 x + c_2, \qquad y' = \tfrac{1}{2}c_1 x^2 + c_2 x + c_3, \qquad y = \tfrac{1}{6}c_1 x^3 + \tfrac{1}{2}c_2 x^2 + c_3 x + c_4,$$

and y is a linear combination of the four functions of the given basis, as wanted.

5. Linear independence. To show that $1, e^{-x}\cos 2x$, and $e^{-x}\sin 2x$ are solutions of $y''' + 2y'' + 5y = 0$ we identify the given ODE to be a *third-order* homogeneous linear ODE with constant coefficients. Therefore, we extend the method of Sec. 2.2 to higher order ODEs and obtain the *third-order* characteristic equation

$$\lambda^3 - 2\lambda^2 + 5\lambda = \lambda(\lambda^2 + 2\lambda + 5) = 0.$$

We immediately see that the first root is $\lambda_1 = 0$. We have to factor the quadratic equation. From elementary algebra we had to memorize that some point that the roots of *any* quadratic equation $ax^2 + bx + c = 0$ are $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, so that here we have, if we use this formula [instead of (4) of Sec. 2.2, p. 54, where a and b are slightly different!],

$$\lambda_{2,3} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4 \cdot 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

Together we obtain roots

$$\lambda_1 = 0, \qquad \lambda_2 = -1 + 2i, \qquad \lambda_3 = -1 - 2i.$$

Noting that $e^{\lambda_1} = e^0 = 1$ and applying the table "Summary of Cases I–III" of Sec. 2.2 twice (!), that is, for Case I ($\lambda_1 = 0$) and for Case III (complex conjugate roots), we see that the solution of the given third-order ODE is

$$y = c_1 \cdot 1 + e^{-x}(A\cos 2x + B\sin 2x) = c_1 \cdot 1 + e^{-x}A\cos 2x + e^{-x}A\sin 2x$$

and that the given functions $1, e^{-x}\cos 2x$, and $e^{-x}\sin 2x$ are solutions to the ODE. The ODE has continuous (even constant) coefficients, so we can apply Theorem 3. To do so, we have to compute the Wronskian as follows:

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & e^{-x}\cos 2x & e^{-x}\sin 2x \\ 0 & y_2' & y_3' \\ 0 & y_2'' & y_3'' \end{vmatrix} \\ &= 1\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - 0\begin{vmatrix} e^{-x}\cos 2x & e^{-x}\sin 2x \\ y_2'' & y_3'' \end{vmatrix} + 0\begin{vmatrix} e^{-x}\cos 2x & e^{-x}\sin 2x \\ y_2' & y_3' \end{vmatrix} \\ &= \begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} = y_2'y_3'' - y_3'y_2''. \end{aligned}$$

Remark on computing higher order determinants. Note that we developed the 3×3 determinant of the Wronskian along the first column, which consisted of mostly zeros. [Note that we always want to "develop" (i.e., break) a big determinant into smaller determinants along any row or column containing the most zeros.] We pull out the entries one at a time along the row or column where we are developing the big determinant. We block out the row and column of any entry that we develop. Thus for an element located in row 2 and column 1 (here $y_1' = 0$) we block out the second row and the first column, obtaining a 2×2 determinant of the form $\begin{vmatrix} y_2 & y_3 \\ y_2'' & y_3'' \end{vmatrix}$ etc. The signs in front of the entries form a checker-board pattern of pluses (+) and minuses (−) as follows:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Our discussion on determinants should suffice for the present situation. For more details, see **Secs. 7.6 and 7.7.**

To continue our computation of the Wronskian, we need y_2', y_2'' and y_3', y_3''. From calculus (chain rule), we get

$$\begin{aligned} y_2' &= -2e^{-x}\sin(2x) - e^{-x}\cos(2x), & y_2'' &= 4e^{-x}\sin(2x) - 3e^{-x}\cos(2x),\\ y_3' &= 2e^{-x}\cos(2x) - e^{-x}\sin(2x), & y_3'' &= -3e^{-x}\sin(2x) - 4^{-x}\cos(2x). \end{aligned}$$

From this, we have

$$\begin{aligned} y_2'y_3'' &= e^{-2x}(6\sin^2(2x) + 4\cos^2(2x) + 11\sin(2x)\cos(2x)),\\ y_2''y_3' &= -e^{-2x}(4\sin^2(2x) + 6\cos^2(2x) - 11\sin(2x)\cos(2x)). \end{aligned}$$

Hence the desired Wronskian is

$$\begin{aligned} W &= y_2'y_3'' - y_3'y_2''\\ &= e^{-2x}(6\sin^2(2x) + 4\cos^2(2x) + 11\sin(2x)\cos(2x))\\ &\quad + e^{-2x}(4\sin^2(2x) + 6\cos^2(2x) - 11\sin(2x)\cos(2x))\\ &= e^{-2x}(10\sin^2(2x) + 10\cos^2(2x)) = 10e^{-2x} \neq 0 \text{ for all } x. \end{aligned}$$

Hence, by Theorem 3, the functions $1, e^{-x}\cos 2x$, and $e^{-x}\sin 2x$ are linearly independent and so they form a basis of solutions of the ODE $y''' + 2y'' + 5y = 0$.

13. Linear independence. Consider $c_1 \sin x + c_2 \cos x + c_3 \sin 2x = 0$ for $x > 0$. For $x = 2\pi$ we have $0 + c_2 \cdot 1 + 0 = 0$, hence $c_2 = 0$. For $x = \frac{1}{2}\pi$ we have $c_1 \cdot 1 + 0 + 0 = 0$, hence $c_1 = 0$. There remains $c_3 \sin 2x = 0$, hence $c_3 = 0$. We conclude that the given functions are linearly independent.

15. Linear dependence can often be shown by using a functional relation. Indeed, in the present problem we have, by formula (22) in Sec. A3.1 of Appendix 3,

$$\sinh 2x = \sinh(x + x) = \sinh x \cosh x + \sinh x \cosh x = 2\sinh x \cosh x.$$

Similarly,

$$\cosh 2x = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x.$$

Finally, using (19), Sec. A3.1,

$$\sinh 2x + \cosh 2x = \cosh^2 x + \sinh^2 x + 2\sinh x \cosh x = (\cosh x + \sinh x)^2 = (e^x)^2 = e^{2x}.$$

Thus

$$\sinh 2x + \cosh 2x = e^{2x} \quad \text{so} \quad \sinh 2x + \cosh 2x - e^{2x} = 0.$$

(We invite you to browse Sec. A3.1 of Appendix 3 and notice many useful formulas for sin, cos, sinh, cosh, etc.) Putting it all together you have linear **dependence** of $\{\sinh 2x, \cosh 2x, e^{2x}\}$ since the last equation is of the form

$$c_1 \sinh 2x + c_2 \cosh 2x + c_3 e^{2x} = 0$$

and has a **nontrivial** (i.e., nonzero) solution, as just proven.

$$c_1 = 1, \qquad c_2 = 1, \qquad c_3 = -1.$$

Sec. 3.2 Homogeneous Linear ODEs with Constant Coefficients

To get a good understanding of this section, you may want to review **Sec. 2.2** as this section is a nice extension of the material in Sec. 2.2. Higher order homogeneous linear ODEs with constant coefficients are solved by first writing down the characteristic equation (2) (on p. 112) and then factoring this equation. *Often you can guess the value of one root.* Consider the following **new example.** Let's say we want to solve the third-order ODE

$$y''' - 3y'' - 4y' + 6y = 0.$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 - 4\lambda + 6 = 0.$$

Stop for a minute and see whether you can guess a root. If you substitute $\lambda_1 = 1$, you see that $(1)^3 + 3 \cdot (1)^2 - 4 \cdot 1 + 6 = 0$. Voila we have one root. Hence we know that $(\lambda - 1)$ is a factor. We apply long division and get

$$\begin{array}{rl}
\left(\lambda^3 - 3\lambda^2 - 4\lambda + 6\right) & / \ (\lambda - 1) = \lambda^2 - 2\lambda - 6 = 0. \\
-\left(\lambda^3 - \lambda^2\right) & \\
\hline
-2\lambda^2 - 4\lambda + 6 & \\
-\left(-2\lambda^2 + 2\lambda\right) & \\
\hline
-6\lambda + 6 & \\
-(-6\lambda + 6) & \\
\hline
0 &
\end{array}$$

Then we apply our well-known root formula for quadratic equations with $a = 1$, $b = -2$, $c = -6$, and get

$$\lambda_{2,3} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-6)}}{2} = \frac{-2 \pm \sqrt{28}}{2} = \frac{-2 \pm 2\sqrt{7}}{2} = -1 \pm \sqrt{7}.$$

Factoring the characteristic equation was the hardest part of the problem. *If you cannot guess any root, apply a numeric method for solving equations from Sec. 19.2.* Governed by the type of roots that the characteristic equation can have, that is, distinct real roots, simple complex roots, multiple real roots, and multiple complex roots, the solution is determined. This is an extension of the method of Sec. 2.2. In particular you apply the table "Summary of Cases I–III" on p. 58 several times (at least twice! as the order of the ODEs considered are greater than 2!). Going back to our example, we have three distinct real roots and looking at the table on p. 58 (extended) Case I, the basis of the solution is $e^{\lambda_1} = e^x$, $e^{\lambda_2} = e^{(1+\sqrt{7})x}$, $e^{\lambda_3} = e^{(1-\sqrt{7})x}$, so that our final answer is

$$y = c_1 e^x + c_2 e^{(1+\sqrt{7})x} + c_3 e^{(1-\sqrt{7})x}.$$

Problem Set 3.2. Page 116

1. General solution of higher order ODE. From the given linear ODE $y''' + 25y' = 0$, we get that the characteristic equation is

$$\lambda^3 - 25\lambda = \lambda(\lambda^2 + 25) = 0.$$

Its three roots are $\lambda_1 = 1$ (distinct real), $\lambda_{2,3} = \pm 5i$ (complex conjugate). Hence a basis for the solutions is

$$1, \cos 5x, \sin 5x \qquad \text{(by the table on p. 58 in Sec. 2.2, Case I, Case III).}$$

(Note that since $\lambda_{2,3} = \pm 5i = 0 \pm 5i$, so that in the table on p. 58, the factor $e^{-ax/2} = e^0 = 1$, and thus we obtain $e^0 \cos 5x = \cos 5x$, etc.).

5. General solution. Fourth-order ODE. The given homogeneous ODE $(D^4 + 10D^2 + 9I)y = 0$ can be written as

$$y^{iv} + 10y'' + 9y = 0.$$

(Recall from Sec. 2.3 that $D^4 y = y^{iv}$, etc. with D being the differential operator.) The characteristic equation is

$$\lambda^4 + 10\lambda^2 + 9 = 0.$$

To find the roots of this fourth-order equation, we use the idea of Example 3 on p. 107. We set $\lambda^2 = \mu$ in the characteristic equation and get a quadratic(!) equation

$$\mu^2 + 10\mu + 9 = 0,$$

which we know how to factor

$$\mu^2 + 10\mu + 9 = (\mu + 1)(\mu + 9) = 0.$$

(If you don't see it, you use the formula for the quadratic equation with $a = 1$, $b = 10$, $c = 9$). Now $\mu_{1,2} = -1, -9$, so that

$$\lambda^2 = -1,\ \lambda_{1,2} = \pm\sqrt{-1} = \pm i, \qquad \lambda^2 = -9,\ \lambda_{3,4} = \pm\sqrt{-9} = \pm 3i.$$

Accordingly we have two sets of complex conjugate roots (extended Case III of table on p. 58 in Sec. 2.2) and the general solution is

$$y = A_1 \cos x + B_1 \sin x + A_2 \cos 3x + B_2 \sin 3x.$$

13. Initial value problem. The roots of the characteristic equation obtained by a root-finding method (Sec. 19.2) are 0.25, -0.7, and $\pm 0.1i$. From this you can write down the corresponding real general solution

$$y = c_1 e^{0.25x} + c_2 e^{-0.7x} + c_3 \cos 0.1x + c_4 \sin 0.1x.$$

To take care of the initial conditions, you need y', y'', y''', and the values of these derivatives at $x = 0$. Now the differentiations are simple, and when you set $x = 0$, you get 1 for the exponential functions, 1 for the cosine, and 0 for the sine. You equate the values of these derivatives to the proper initial value that is given. This gives you the linear system

$$\begin{aligned}
y(0) &= c_1 + c_2 + c_3 &&= 17.4,\\
y'(0) &= 0.25c_1 - 0.7c_2 + 0.1c_4 &&= -2.82,\\
y''(0) &= 0.0625c_1 + 0.49c_2 - 0.01c_3 &&= 2.0485,\\
y'''(0) &= 0.015625c_1 - 0.343c_2 - 0.001c_4 &&= -1.458675.
\end{aligned}$$

The solution of this linear system is $c_1 = 1$, $c_2 = 4.3$, $c_3 = 12.1$, $c_4 = -0.6$. This gives the particular solution (cf. p. A10)

$$y = e^{0.25x} + 4.3e^{-0.7x} + 12.1 \cos 0.1x - 0.6 \sin 0.1x$$

satisfying the given initial conditions.

Sec. 3.3 Nonhomogeneous Linear ODEs

This section extends the method of undetermined coefficients and the method of variation of parameters to higher order ODEs. You may want to review **Sec. 2.7** and **Sec. 2.10.** One subtle point to consider is the new Modification Rule for higher order ODEs.

Modification Rule. For an ODE of order $n = 2$ we had in Sec. 2.7 either $k = 1$ (single root) or $k = 2$ (double root) and multiplied the choice function by x or x^2, respectively. For instance, if $\lambda = 1$ is a double root ($k = 2$), then e^x and xe^x are solutions, and if the right side is e^x, the choice function is $Cx^k e^x = Cx^2 e^x$ (instead of Ce^x) and is no longer a solution of the homogeneous ODE. Similarly here, for a triple root 1, say, you have solutions e^x, xe^x, x^2e^x, you multiply your choice function Ce^x by $x^k = x^3$, obtaining Cx^3e^x, which is no longer a solution of the homogeneous ODE. This should help you understand the new Modification Rule on p. 117 for higher order ODEs.

Problem Set 3.3. Page 122

1. General solution. Method of undetermined coefficients. The given ODE is

$$y''' + 3y'' + 3y' + y = e^x - x - 1.$$

You must first solve the homogeneous ODE

$$y''' + 3y'' + 3y' + y = 0$$

to find out whether the Modification Rule applies. If you forget this, you will not be able to determine the constants in your choice function.

The characteristic function of the homogeneous ODE is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0.$$

One of the roots is -1. Hence we use long division to find the other roots.

$$\begin{array}{l}
\left(\lambda^3 + 3\lambda^2 + 3\lambda + 1\right) / (\lambda + 1) = \lambda^2 + 2\lambda + 1 = 0. \\
\underline{-\left(\lambda^3 + \lambda^2\right)\qquad\;} \\
\qquad 2\lambda^2 + 3\lambda \\
\qquad \underline{-\left(2\lambda^2 + 2\lambda\right)\quad} \\
\qquad\qquad\qquad \lambda + 1 \\
\qquad\qquad \underline{-(\lambda + 1)} \\
\qquad\qquad\qquad\quad 0
\end{array}$$

Now $\lambda^2 + 2\lambda + 1 = (\lambda + 1)(\lambda + 1)$, thus

$$(\lambda^3 + 3\lambda^2 + 3\lambda + 1) = (\lambda^2 + 2\lambda + 1)(\lambda + 1) = (\lambda + 1)(\lambda + 1)(\lambda + 1) = (\lambda + 1)^3.$$

Hence you have a real triple root. By Case II (extended), table on p. 58 and (7) of Sec. 3.2, you obtain the general solution y_h of the homogeneous ODE

$$y_h = (c_1 + c_2x + c_3x^2)e^{-x}.$$

Now determine a particular solution of the nonhomogeneous ODE. You see that no term on the right side of the given ODE is a solution of the homogeneous ODE, so that for $r = r_1 + r_2 = e^{-x} - x - 1$ you can choose

$$y_p = y_{p_1} + y_{p_2} = Ce^x + (K_1x + K_0).$$

Differentiate this three times to get

$$\begin{aligned} y_p' &= \qquad Ce^x + K_1, \\ y_p'' &= \qquad Ce^x, \\ y_p''' &= \qquad Ce^x. \end{aligned}$$

Substitute these expressions into the given ODE, obtaining

$$Ce^x + 3Ce^x + 3(Ce^x + K_1) + Ce^x + (K_1x + K_0) = e^{-x} - x - 1.$$

Equate the exponential terms on both sides,

$$C(1 + 3 + 3 + 1) = 8C = 1, \quad C = \tfrac{1}{8}.$$

By equating the x-, and x^0-terms on both sides you find

$$\begin{aligned} K_1 &= -1, \\ 3K_1 + K_0 = -1, \qquad K_0 &= -1 - 3K_1 = -1 - 3(-1) = 2. \end{aligned}$$

You thus obtain

$$y_p = \tfrac{1}{8}e^x - x + 2.$$

Together with the above y_h, you have the general solution of the nonhomogeneous ODE given on p. A9.

Remark. Note that you don't need a third term r_3 when you set up the start of finding the particular solution because r_2 took care of both $-x$ and -1.

13. Initial value problem. Method of variation of parameters. If you had trouble solving this problem, let me give you a hint: Take a look at the integration formulas in the front of the book. The given ODE (notation from Sec. 2.3, p. 60) is

$$y''' + 4y' = 10\cos x + 5\sin x.$$

The auxiliary equation of the homogeneous ODE is

$$\lambda^3 - 4\lambda = \lambda(\lambda^2 - 4) = \lambda(\lambda + 2)(\lambda - 2).$$

The roots are 0, -2, and $+2$. You thus have the basis of solutions

$$y_1 = 1, \quad y_2 = e^{-2x}, \quad y_3 = e^{2x},$$

and a corresponding general solution of the homogeneous ODE

$$y_h = c_1 + c_2e^{-2x} + c_3e^{2x}.$$

Determine a particular solution by variation of parameters according to formula (7) on p. 118. From the right side of the ODE you get

$$r = 10\cos x + 5\sin x.$$

In (7) you also need the Wronskian W and W_1, W_2, W_3, which look similar to the determinants on p. 119 and have the values

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & e^{-2x} & e^{2x} \\ 0 & -2e^{-2x} & 2e^{2x} \\ 0 & 4e^{-2x} & 4e^{2x} \end{vmatrix}$$
$$= 1 \cdot \begin{vmatrix} -2e^{-2x} & 2e^{2x} \\ 4e^{-2x} & 4e^{2x} \end{vmatrix} = -8e^0 - 8e^0 = -16,$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & e^{-2x} & e^{2x} \\ 0 & -2e^{-2x} & 2e^{2x} \\ 1 & 4e^{-2x} & 4e^{2x} \end{vmatrix}$$
$$= 1 \cdot \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 2 - (-2) = 4,$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & 0 & e^{2x} \\ 0 & 0 & 2e^{2x} \\ 0 & 1 & 4e^{2x} \end{vmatrix}$$
$$= 1 \cdot \begin{vmatrix} 0 & 2e^{2x} \\ 1 & 4e^{2x} \end{vmatrix} = 0 - 2e^{2x} = -2e^{2x},$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} = \begin{vmatrix} 1 & e^{-2x} & 0 \\ 0 & -2e^{-2x} & 0 \\ 0 & 4e^{-2x} & 1 \end{vmatrix}$$
$$= 1 \cdot \begin{vmatrix} -2e^{-2x} & 0 \\ 4e^{-2x} & 1 \end{vmatrix} = -2e^{-2x}.$$

We solve the following equation from (7), p. 118:

$$y_p = y_1(x) \int \frac{W_1(x)}{W(x)} r(x)\, dx + y_2(x) \int \frac{W_2(x)}{W(x)} r(x)\, dx + y_3(x) \int \frac{W_3(x)}{W(x)} r(x)\, dx$$
$$= 1 \cdot \int \frac{4}{-16}(10 \cos x + 5 \sin x)\, dx + e^{-2x} \int \frac{-2e^{2x}}{-16}(10 \cos x + 5 \sin x)\, dx$$
$$+ e^{2x} \int \frac{-2e^{-2x}}{-16}(10 \cos x + 5 \sin x)\, dx.$$

To do this evaluation you may want to use these integration formulas (where a is a constant)

$$\int e^{ax} \cos x\, dx = \frac{1}{a^2+1} e^{ax}(a \cos x + \sin x), \qquad \int e^{ax} \sin x\, dx = \frac{1}{a^2+1} e^{ax}(a \sin x - \cos x).$$

(Note that the two integral formulas are variations of the last two equations in the front of the book on Integration with $a = b$.) Using this hint, evaluate these three integrals by breaking them into six integrals, using the basic rule $\int (af + bg)\, dx = a \int f\, dx + b \int g\, dx$ (a, b constants), evaluating each of them separately. Finally, you multiply your results by the functions in front of them (those solutions y_1, y_2, y_3). You should obtain

$$y_p = \tfrac{5}{4}(\cos x - 2\sin x) + e^{-2x}\left(\tfrac{1}{8}e^{2x}(4\sin x + 3\cos x)\right) - e^{2x}\left(\tfrac{5}{8}e^{-2x}\cos x\right)$$
$$= \cos x\Big(\tfrac{10}{8} + \tfrac{3}{8} - \tfrac{5}{8}\Big) + \sin x\Big(-\tfrac{10}{4} + \tfrac{4}{8}\Big) = \cos x - 2\sin x.$$

Putting it together, the general solution for the given ODE is

$$y(x) = y_h(x) + y_p(x) = (c_1 + c_2 x + c_3 x^2)e^{-x} + \cos x - 2\sin x.$$

Since this is an initial value problem, we continue. The first initial condition $y(0) = 3$ can be immediately plugged into the general solution. We get

$$y(0) = c_1 + c_2 + c_3 + 1 - 2 \cdot 0 = 3.$$

This gives us

$$\boxed{c_1 + c_2 + c_3 = 2}$$

Next for the second initial condition $y'(0) = -2$, we need the first derivative

$$y'(x) = -2c_1 e^{-2x} + 2c_3 e^{2x} - \sin x - 2\cos x.$$

Calculate $y'(0)$ and simplify to obtain

$$y'(0) = -2c_1 + 2c_3 - 0 - 2 = -2.$$
$$\boxed{-2c_2 + 2c_3 = 0.}$$

Finally, repeating this step for the third initial condition $y''(0) = -1$ we have

$$y''(x) = 4c_2 e^{-2x} + 4c_3 e^{2x} - \cos x + 2\sin x.$$
$$y''(0) = 4c_2 + 4c_3 - 1 + 0 = -1.$$
$$\boxed{4c_2 + 4c_3 = 0.}$$

The three boxed equations gives us a system of linear equations. If it were more complicated, we could use Gauss elimination or Cramer's rule (see Sec. 7.6 for reference). But here we can just do some substitutions as follows. From the third boxed equation we get that $c_2 = -c_3$. Substitute this into the second boxed equation and obtain $-2(-c_3) + 2c_3 = 0$, so that $c_3 = 0$. From this we immediately get $c_2 = -c_3 = 0$, which means $c_2 = 0$. Putting both $c_2 = 0$ and $c_3 = 0$ into the first boxed equation, we obtain $c_1 = 2$. Thus we have

$$c_1 = 2, \qquad c_2 = 0, \qquad c_3 = 0.$$

Thus we have solved the IVP. Hence the answer to the IVP is (cf. p. A9)

$$y(x) = 2 + \cos x - 2\sin x.$$

Chap. 4 Systems of ODEs. Phase Plane. Qualitative Methods

The methods discussed in this chapter use elementary linear algebra (Sec. 4.0). First we present another method for solving higher order ODEs that is different from the methods of Chap. 3. This method consists of converting any nth-order ODE into a system of n first-order ODEs, and then solving the system obtained (Secs. 4.1–4.4 for homogeneous linear systems). We also discuss a totally new way of looking at systems of ODEs, that is, a *qualitative* approach. Here we want to know the *behavior* of families of solutions of ODEs *without actually solving* these ODEs. This is an attractive method for nonlinear systems that are difficult to solve and can be approximated by linear systems by removing nonlinear terms. This is called **linearization** (Sec. 4.5). In the last section we solve nonlinear systems by the method of undetermined coefficients, a method you have seen before in Secs. 2.7 and 2.10.

Sec. 4.0 For Reference: Basics of Matrices and Vectors

This section reviews the basics of linear algebra. Take a careful look at **Example 1** on p. 130. For this chapter you have to know how to calculate the **characteristic equation** of a square 2×2 (or at most a 3×3) matrix and how to determine its eigenvalues and eigenvectors. To obtain the determinant of $\mathbf{A} - \lambda\mathbf{I}$, denoted by $\det(\mathbf{A} - \lambda\mathbf{I})$, you first have to compute $\mathbf{A} - \lambda\mathbf{I}$. Note that λ is a scalar, that is, a number. In the following calculation, the second equality holds because of scalar multiplication and the third equality by matrix addition:

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$
$$= \begin{bmatrix} -4.0 - \lambda & 4.0 - 0 \\ -1.6 - 0 & 1.2 - \lambda \end{bmatrix} = \begin{bmatrix} -4.0 - \lambda & 4.0 \\ -1.6 & 1.2 - \lambda \end{bmatrix}.$$

Then you compute (see solution to Prob. 5 of Problem Set 3.1, p. 37, on how to calculate **determinants**)

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -4.0 - \lambda & 4.0 \\ -1.6 & 1.2 - \lambda \end{vmatrix} = \lambda^2 + 2.8\lambda + 1.6$$
$$= (\lambda + 2)(\lambda + 0.8) = 0.$$

The roots of the characteristic polynomial are called the **eigenvalues** of matrix $\mathbf{A}$. Here they are $\lambda_1 = -2$ and $\lambda_2 = 0.8$. The calculations for the eigenvector corresponding to $\lambda_1 = -2$ are shown in the book. To determine an eigenvector corresponding to $\lambda_2 = 0.8$, you first have to substitute $\lambda = 0.8$ into the system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \begin{bmatrix} -4.0 - \lambda & 4.0 \\ -1.6 & 1.2 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (-4.0 - \lambda)x_1 + 4.0x_2 \\ -1.6x_1 + (-1.2 - \lambda)x_2 \end{bmatrix} = \mathbf{0}.$$

This gives

$$\begin{bmatrix} (-4.0 - 0.8)x_1 + 4.0x_2 \\ -1.6x_1 + (-1.2 - 0.8)x_2 \end{bmatrix} = \begin{bmatrix} -4.8x_1 + 4.0x_2 \\ -1.6x_1 + 2.0x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Using the second equation, that is, $-1.6x_1 + 2.0x_2 = 0$, and setting $x_1 = 1.0$, gives $2.0x_2 = 1.6$, so that $x_2 = 1.6/2.0 = 0.8$. Thus an **eigenvector corresponding to the eigenvalue** $\lambda_2 = 0.8$ is $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$. *Also note that eigenvectors are only determined up to a nonzero constant*. We could have chosen $x_1 = 3.0$ and gotten $x_2 = 4.8/2.0 = 2.4$, thereby obtaining an eigenvector $\begin{bmatrix} 3 \\ 2.4 \end{bmatrix}$ corresponding to λ_2.

Sec. 4.1 Systems of ODEs as Models in Engineering Applications

The most important idea of this section is Theorem 1 on p. 135 and applied in Example 3. You will convert second-order ODEs into a system of two first-order ODEs. Examples 1 and 2 show how, by growing a problem from *one* tank to *two* tanks or from *one* circuit to *two* circuits, the number of ODEs grows accordingly. This is attractive mathematics as explained in the Remark on p. 134.

Example 2. Electrical network. Spend time on Fig. 80 on p. 134, until you feel that you fully understand the difference between (a) and (b). Figure 80a represents the vector solution to the problem as two separate components as you are used to in calculus. Figure 80b gives a parametric representation and introduces a **trajectory**. Trajectories will play an important role throughout this chapter. Try to understand the behavior of the trajectory and gain a more qualitative understanding of the solution. The trajectory starts at the origin. It reaches its highest point where I_2 has a maximum (before $t = 1$). It has a vertical tangent where I_1 has a maximum, shortly after $t = 1$. As t increases from there to $t = 5$, the trajectory goes downward until it almost reaches the I_1-axis at 3; this point is a limit as $t \to \infty$. In terms of t the trajectory goes up faster than it comes down.

Problem Set 4.1. Page 136

7. Electrical network. The problem amounts to the determination of the two arbitrary constants in a general solution of a system of two ODEs in two unknown functions I_1 and I_2, representing the currents in an electrical network shown in Fig. 79 in Sec. 4.1. You will see that this is quite similar to the corresponding task for a single second-order ODE. That solution is given by (6), in components

$$I_1(t) = 2c_1e^{-2t} + c_2e^{-0.8t} + 3, \qquad I_2(t) = c_1e^{-2t} + 0.8c_2e^{-0.8t}.$$

Setting $t = 0$ and using the given initial conditions $I_1(0) = 0$, $I_2(0) = -3$ gives two equations

$$\begin{aligned} I_1(0) &= 2c_1 + c_2 + 3 = 0 \\ I_2(0) &= c_1 + 0.8c_2 = -3. \end{aligned}$$

From the first equation you have $c_2 = -3 - 2c_1$. Substituting this into the second equation lets you determine the value for c_1, that is,

$$c_1 + 0.8(-3 - 2c_1) = -0.6c_1 - 2.4 = -3, \quad \text{hence } c_1 = 1.$$

Also $c_2 = -3 - 2c_1 = -3 - 2 = -5$. This yields the answer

$$\begin{aligned} I_1(t) &= 2e^{-2t} - 5e^{-0.8t} + 3 \\ I_2(t) &= e^{-2t} + 0.8(-5)e^{-0.8t} = e^{-2t} - 4e^{-0.8t}. \end{aligned}$$

You see that the limits are 3 and 0, respectively. Can you see this directly from Fig. 79 for physical reasons?

13. Conversion of a single ODE to a system. This conversion is an important process, which always follows the pattern shown in formulas (9) and (10) of Sec. 4.1. The present equation $y'' + 2y' - 24y = 0$ can be readily solved as follows. Its characteristic equation (directly from Sec. 2.2) is $\lambda^2 + 2\lambda - 24 = (\lambda - 4)(\lambda + 6)$. It has roots 4, -6 so that the general solution of the ODE is $y = c_1e^{4t} + c_2e^{-6t}$. The point of the problem is to explain the relation between systems of ODEs and single ODEs and their solutions. To explore this new idea, we chose a simple problem, whose solution can be readily obtained. (Thus we are not trying to explain a more complicated method for a simple problem!) In the present case the formulas (9) and (10) give $y_1 = y$, $y_2 = y'$ and

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 24y_1 - 2y_2 \end{aligned}$$

(because the given equation can be written $y'' = 24y - 2y'$, hence $y_1'' = 24y_1 - 2y_2$, but $y_1'' = y_2'$). In matrix form (as in Example 3 of the text) this is

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ 24 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Then you compute

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 0 & 1 \\ 24 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 24 & -2-\lambda \end{bmatrix}.$$

Then the characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 24 = (\lambda - 4)(\lambda + 6) = 0.$$

The eigenvalues, which are the roots of the characteristic equation, are $\lambda_1 = 4$ and $\lambda_2 = -6$. For λ_1 you obtain an eigenvector from (13) in Sec. 4.0 with $\lambda = \lambda_1$, that is,

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \begin{bmatrix} -4 & 1 \\ 24 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_1 + x_2 \\ 24x_1 - 6x_2 \end{bmatrix} = \mathbf{0}.$$

From the first equation $-4x_1 + x_2 = 0$ you have $x_2 = 4x_1$. An eigenvector is determined only up to a nonzero constant. Hence, in the present case, a convenient choice is $x_1 = 1$, which, when substituted into the first equation, gives $x_2 = 4$. Thus an eigenvector corresponding to $\lambda_1 = 4$ is

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

(The second equation gives the same result and is not needed.) For the second eigenvalue, $\lambda_2 = -6$, you proceed the same way, that is,

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x} = \begin{bmatrix} 6 & 1 \\ 24 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 + x_2 \\ 24x_1 + 4x_2 \end{bmatrix} = \mathbf{0}.$$

You now have $6x_1 + x_2 = 0$, hence $x_2 = -6x_1$, and can choose $x_1 = 1$, thus obtaining $x_2 = -6$. Thus an eigenvector corresponding to $\lambda_2 = -6$ is

$$\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$$

Expressing the two eigenvectors in transpose (T) notation, you have

$$\mathbf{x}^{(1)} = [1 \quad 4]^{\mathsf{T}} \quad \text{and} \quad \mathbf{x}^{(2)} = [1 \quad -6]^{\mathsf{T}}.$$

Multiplying these by e^{4t} and e^{-6t}, respectively, and taking a linear combination involving two arbitrary constants c_1 and c_2 gives a general solution of the present system in the form

$$\mathbf{y} = c_1[1 \quad 4]^{\mathsf{T}} e^{4t} + c_2[1 \quad -6]^{\mathsf{T}} e^{-6t}.$$

In components, this is, corresponding to the answer on p. A9

$$y_1 = c_1 e^{4t} + c_2 e^{-6t}$$
$$y_2 = 4c_1 e^{4t} - 6c_2 e^{-6t}.$$

Here you see that $y_1 = y$ is a general solution of the given ODE, and $y_2 = y_1' = y'$ is the derivative of this solution, as had to be expected because of the definition of y_2 at the beginning of the process.

Note that you can use $y_2 = y_1'$ for checking your result.

Sec. 4.2 Basic Theory of Systems of ODEs. Wronskian

The ideas are similar to those of Secs. 1.7 and 2.6. You should know what a Wronskian is and how to compute it. The theory has no surprises and you will use it naturally as you do your homework exercises.

Sec. 4.3 Constant-Coefficient Systems. Phase Plane Method

In this section we study the phase portrait and show five types of critical points. They are **improper nodes** (Example 1, pp. 141–142, Fig. 82), **proper nodes** (Example 2, Fig. 83, p. 143), **saddle points** (Example 3, pp. 143–144, Fig. 84), **centers** (Example 4, p. 144, Fig. 85), and **spiral points** (Example 5, pp. 144–145, Fig. 86). There is also the possibility of a **degenerate node** as explained in Example 6 and shown in Fig. 87 on p. 146.

Example 2. Details are as follows. The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 = 0.$$

Thus $\lambda = 1$ is an eigenvalue. Any nonzero vector with two components is an eigenvector because $\mathbf{Ax} = \mathbf{x}$ for any $\mathbf{x}$; indeed, $\mathbf{A}$ is the 2×2 unit matrix! Hence you can take $\mathbf{x}^{(1)} = [1 \quad 0]^\mathsf{T}$ and $\mathbf{x}^{(2)} = [0 \quad 1]^\mathsf{T}$ or any other two linearly independent vectors with two components. This gives the solution on p. 143.

Example 3. $(1-\lambda)(-1-\lambda) = (\lambda - 1)(\lambda + 1) = 0$, and so on.

Problem Set 4.3. Page 147

1. General solution. The matrix of the system $y_1' = y_1 + y_2$, $y_2' = 3y_1 - y_2$ is

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}.$$

From this you have the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 3 = \lambda^2 - 4 = (\lambda + 2)(\lambda - 2) = 0.$$

You see that the eigenvalues are ± 2. You obtain eigenvectors for -2 from $3x_1 + x_2 = 0$, say $x_1 = 1$, $x_2 = -3$ (recalling that eigenvectors are determined only up to an arbitrary nonzero factor). Thus for $\lambda = -2$ you have an eigenvector of $[1 \quad -3]^{\mathsf{T}}$. Similarly, for $\lambda = 2$ you obtain an eigenvector from $-x_1 + x_2 = 0$, say, $[1 \quad 1]^{\mathsf{T}}$. You thus obtain the general solution

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

On p. A10 this is written in terms of components.

7. General solution. Complex eigenvalues. Write down the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Then calculate

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 0 & 0 \\ -1 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{bmatrix}.$$

From this we obtain the characteristic equation by taking the determinant of $\mathbf{A} - \lambda\mathbf{I}$, that is,

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 & 0 \\ -1 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} &= -\lambda \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & 1 \\ 0 & -\lambda \end{vmatrix} \\ &= -\lambda(\lambda^2 + 1) - (\lambda) = -\lambda^3 - \lambda - \lambda = -(\lambda^3 + 2\lambda) \\ &= -\lambda(\lambda^2 + 2) = -\lambda(\lambda - \sqrt{2}i)(\lambda + \sqrt{2}i) = 0. \end{aligned}$$

The roots of the characteristic equation are $0, \pm\sqrt{2}i$. Thus the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -\sqrt{2}i$, and $\lambda_3 = +\sqrt{2}i$.

For $\lambda_1 = 0$ we obtain an eigenvector from

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives us a system consisting of 3 homogeneous linear equations, that is

$$\begin{aligned} x_2 \qquad &= 0. \\ -x_1 \qquad + x_3 &= 0 \quad \text{so that} \quad x_3 = x_1. \\ -x_2 \qquad &= 0 \end{aligned}$$

Thus, if we choose $x_1 = 1$, then $x_3 = 1$. Also $x_2 = 0$ from the first equation. Thus $[1 \quad 0 \quad 1]^{\mathsf{T}}$ is an eigenvector for $\lambda_1 = 0$.

For $\lambda_2 = -\sqrt{2}i$, we obtain an eigenvector as follows:

$$\begin{bmatrix} \sqrt{2}i & 1 & 0 \\ -1 & \sqrt{2}i & 1 \\ 0 & -1 & \sqrt{2}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the following system of linear equations:

$$\begin{aligned} \sqrt{2}ix_1 + \quad x_2 \qquad\quad &= 0 \quad \text{so that} \quad x_2 = -\sqrt{2}ix_1. \\ -x_1 + \sqrt{2}x_2 + \quad x_3 &= 0, \\ -x_2 + \sqrt{2}ix_3 &= 0. \end{aligned}$$

Substituting $x_3 = \sqrt{2}ix_1$ (obtained from the first equation) into the second equation gives us

$$\begin{aligned} -x_1 + \sqrt{2}x_2 + x_3 &= -x_1 + (\sqrt{2}i)(-\sqrt{2}i)x_1 + x_3 \\ &= -x_1 + 2x_1 + x_3 = x_1 + x_3 = 0 \quad \text{hence} \quad x_1 = -x_3. \end{aligned}$$

[Note that, to simplify the coefficient of the x_1-term, we used that $(\sqrt{2}i)(-\sqrt{2}i) = -(\sqrt{2})(\sqrt{2}) \cdot (\sqrt{-1})(\sqrt{-1}) = -(2)(-1) = -2$, where $i = \sqrt{-1}$.] Setting $x_1 = 1$ gives $x_3 = -1$, and $x_2 = -\sqrt{2}i$. Thus the eigenvector for $\lambda_2 = -\sqrt{2}i$ is

$$[x_1 \quad x_2 \quad x_3]^{\mathsf{T}} = [1 \quad -\sqrt{2}i \quad -1]^{\mathsf{T}}.$$

For $\lambda_3 = \sqrt{2}i$, we obtain the following system of linear equations:

$$\begin{aligned} -\sqrt{2}ix_1 + \quad x_2 \qquad\quad &= 0 \quad \text{so that} \;\; x_2 = \sqrt{2}ix_1. \\ -x_1 - \sqrt{2}ix_2 + \quad x_3 &= 0, \\ -x_2 - \sqrt{2}ix_3 &= 0 \quad \text{so that} \;\; x_2 = -\sqrt{2}ix_1. \end{aligned}$$

Substituting $x_2 = \sqrt{2}ix_1$ (obtained from the first equation) into the second equation

$$x_1 = -\sqrt{2}x_2 + x_3 = -\sqrt{2}i\sqrt{2}ix_1 + x_3 = 2x_1 + x_3, \;\; \text{hence } x_1 = -x_3.$$

(Another way to see this is to note that, $x_2 = \sqrt{2}ix_1$ and $x_2 = -\sqrt{2}ix_3$, so that $\sqrt{2}ix_1 = -\sqrt{2}ix_3$ and hence $x_1 = -x_3$.) Setting $x_1 = 1$ gives $x_3 = -1$, and $x_2 = \sqrt{2}i$. Thus the eigenvector for $\lambda_3 = \sqrt{2}i$ is $[1 \quad \sqrt{2}i \quad -1]^{\mathsf{T}}$, as was to be expected from before. For more complicated calculations, you might want to use Gaussian elimination (to be discussed in Sec. 7.3).

Together, we obtain the general solution

$$\mathbf{y} = c_1^* \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{0t} + c_2^* \begin{bmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{bmatrix} e^{-\sqrt{2}it} + c_3^* \begin{bmatrix} 1 \\ \sqrt{2}i \\ -1 \end{bmatrix} e^{-\sqrt{2}it}$$

where c_1^*, c_2^*, c_3^* are constants. By use of the Euler formula (see (11) in Sec. 2.2, p. 58)

$$= c_1^* \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2^* \begin{bmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{bmatrix} (\cos\sqrt{2}t - i\sin\sqrt{2}t) + c_3^* \begin{bmatrix} 1 \\ \sqrt{2}i \\ -1 \end{bmatrix} (\cos\sqrt{2}t + i\sin\sqrt{2}t).$$

Write out in components and collect cosine and sine terms—the first component is

$$\begin{aligned} y_1 &= c_1^* + c_2^*(\cos\sqrt{2}t - i\sin\sqrt{2}t) + c_3^*(\cos\sqrt{2}t + i\sin\sqrt{2}t) \\ &= c_1^* + (c_2^* + c_3^*)\cos\sqrt{2}t + i(c_3^* - c_2^*)\sin\sqrt{2}t. \end{aligned}$$

Now set $A = -c_2^* - c_3^*$, $B = i(c_3^* - c_2^*)$ and get

$$y_1 = c_1^* - A\cos\sqrt{2}t + B\sin\sqrt{2}t.$$

Similarly, the second component is

$$\begin{aligned} y_2 &= -\sqrt{2}ic_2^*(\cos\sqrt{2}t - i\sin\sqrt{2}t) + \sqrt{2}ic_3^*(\cos\sqrt{2}t + i\sin\sqrt{2}t) \\ &= -\sqrt{2}ic_2^*\cos\sqrt{2}t - \sqrt{2}ic_2^*\sin\sqrt{2}t + \sqrt{2}ic_3^*\cos\sqrt{2}t - \sqrt{2}ic_3^*\sin\sqrt{2}t \\ &= \cos\sqrt{2}t(-\sqrt{2}ic_2^* + \sqrt{2}ic_3^*) + \sin\sqrt{2}t(-\sqrt{2}ic_2^* - \sqrt{2}ic_3^*) \\ &= \cos\sqrt{2}t\sqrt{2}i(c_3^* - c_2^*) + \sin\sqrt{2}t\sqrt{2}(c_2^* - c_3^*) \\ &= \cos\sqrt{2}t\sqrt{2}\cdot B + \sin\sqrt{2}t\sqrt{2}\cdot A \\ &= A\sqrt{2}\sin\sqrt{2}t + B\sqrt{2}\cos\sqrt{2}t. \end{aligned}$$

For the third component we compute algebraically

$$\begin{aligned} y_3 &= c_1^* - c_2^*(\cos\sqrt{2}t - i\sin\sqrt{2}t) - c_3^*(\cos\sqrt{2}t + i\sin\sqrt{2}t) \\ &= c_1^* - c_2^*\cos\sqrt{2}t + c_2^*i\sin\sqrt{2}t - c_3^*\cos\sqrt{2}t - c_3^*i\sin\sqrt{2}t \\ &= c_1^* + \cos\sqrt{2}t(-c_2^* - c_3^*) + \sin\sqrt{2}ti(c_2^* - c_3^*) \\ &= c_1^* + \cos\sqrt{2}t\cdot A - \sin\sqrt{2}t\cdot B \\ &= c_1^* + A\cos\sqrt{2}t - B\sin\sqrt{2}t. \end{aligned}$$

Together we have

$$\begin{aligned} y_1 &= c_1^* - A\cos\sqrt{2}t + B\sin\sqrt{2}t, \\ y_2 &= A\sqrt{2}\sin\sqrt{2}t + B\sqrt{2}\cos\sqrt{2}t, \\ y_3 &= c_1^* + A\cos\sqrt{2}t - B\sin\sqrt{2}t. \end{aligned}$$

This is precisely the solution given on p. A10 with $c_1^* = c_1$, $A = c_2$, $B = c_3$.

15. Initial value problem. Solving an initial value problem for a system of ODEs is similar to that of solving an initial value problem for a single ODE. Namely, you first have to find a general solution and then determine the arbitrary constants in that solution from the given initial conditions.

To solve Prob. 15, that is,

$$\begin{aligned} y_1' &= 3y_1 + 2y_2, \\ y_2' &= 2y_1 + 3y_2, \\ y_1(0) = 0.5, &\qquad y_2(0) = 0.5, \end{aligned}$$

write down the matrix of the system

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$

Then

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

and determine the eigenvalues and eigenvectors as before. Solve the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda) - 4 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5) = 0.$$

You see that the eigenvalues are $\lambda = 1$ and 5. For $\lambda = 1$ obtain an eigenvector from $(3-1)x_1 + 2x_2 = 0$, say, $x_1 = 1$, $x_2 = -2$. Similarly, for $\lambda = 5$ obtain an eigenvector from $(3-5)\,x_1 + 2x_2 = 0$, say, $x_1 = 2$, $x_2 = 1$. You thus obtain the general solution

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}.$$

From this, and the initial conditions, you have, setting $t = 0$,

$$\mathbf{y}(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}.$$

From the second component you obtain $-c_1 + c_2 = -0.5$, hence $c_2 = -0.5 + c_1$. From this, and the first component, you obtain

$$c_1 + c_2 = c_1 - 0.5 + c_1 = 0.5, \quad \text{hence } c_1 = 0.5.$$

Conclude that $c_2 = -0.5 + c_1 = -0.5 + 0.5 = 0$ and get, as on p. A10,

$$\mathbf{y} = 0.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t = \begin{bmatrix} 0.5e^t \\ -0.5e^t \end{bmatrix}.$$

Written in components, this is

$$\begin{aligned} y_1 &= 0.5e^t, \\ y_2 &= -0.5e^t. \end{aligned}$$

Sec. 4.4 Criteria for Critical Points. Stability

The type of critical point is determined by quantities closely related to the eigenvalues of the matrix of the system, namely, the trace p, which is the sum of the eigenvalues, the determinant q, which is the product of the eigenvalues, and the discriminant Δ, which equals $p^2 - 4q$; see (5) on p. 148. Whereas, in Sec. 4.3, we used the phase portrait to graph critical points, here we use algebraic criteria to identify critical points. Table 4.1 (p. 149) is important in identification. Table 4.2 (p. 150) gives different types of stability. You will use both tables in the problem set.

Problem Set 4.4. Page 151

7. Saddle point. We are given the system

$$y_1' = y_1 + 2y_2,$$
$$y_2' = 2y_1 + y_2.$$

From the matrix of the system

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

From (5) of Sec. 4.4, you get $p = 1 + 1 = 2$, $q = 1^2 - 2^2 = -3$, and $\Delta = p - 4q = 2^2 - 4(-3) = 4 + 12 = 16$. Since $q < 0$, we have a saddle point at $(0, 0)$. Indeed, the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 \\ 8 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0$$

has the real roots -1, 3, which have opposite signs, as it should be according to Table 4.1 on p. 149. Also, $q < 0$ implies that the critical point is unstable. Indeed, saddle points are always unstable.

To find a general solution, determine eigenvectors. For $\lambda = -1$ you find an eigenvector from $(1 - \lambda)x_1 + 2x_2 = 2x_1 + 2x_2 = 0$, say, $x_1 = 1$, $x_2 = -1$, giving $[1 \quad -1]^\mathsf{T}$. Similarly, for $\lambda = 3$ you have $-2x_1 + 2x_2 = 0$, say, $x_1 = 1$, $x_2 = 1$, so that an eigenvector is $[1 \quad 1]^\mathsf{T}$. You thus obtain the general solution

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

and in components is

$$y_1 = c_1 e^{-t} + c_2 e^{3t},$$
$$y_2 = -c_1 e^{-t} + c_2 e^{3t}.$$

11. Damped oscillations. The ODE $y'' + 2y' + 2y = 0$ has the characteristic equation

$$\lambda^2 + 2\lambda + 2 = (\lambda + 1 + i)(\lambda + 1 - i) = 0.$$

You thus obtain the roots $-1 - i$ and $-1 + i$ and the corresponding real general solution (Case III, complex conjugate, Sec. 2.2)

$$y = e^{-t}(A \cos t + B \sin t)$$

(see p. A10). This represents a damped oscillation.

Convert this to a system of ODEs

$$y_1' = y_2$$
$$y'' = y_1'' = y_2' = -2y_1 - 2y_2.$$

Write this in matrix form,

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Hence

$$\begin{aligned} p &= \lambda_1 + \lambda_2 = (-1 - i) + (-1 + i) = -2, \\ q &= \lambda_1 \lambda_2 = (-1 - i)(-1 + i) = 1 - i^2 = 2, \\ \Delta &= (\lambda_1 - \lambda_2)^2 = (-2i)^2 = -4. \end{aligned}$$

Since $p \neq 0$ and $\Delta < 0$, we have spirals by Table 4.1(d). Furthermore, these spirals are stable and attractive by Table 4.2(a).

Since the physical system has damping, energy is taken from it all the time, so that the motion eventually comes to rest at (0, 0).

17. Perturbation. If the entries of the matrix of a system of ODEs are measured or recorded values, errors of measurement can change the type of the critical point and thus the entire behavior of the system.

The unperturbed system

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}$$

has a center by Table 4.1(c). Now Table 4.1(d) shows that a slight change of p (which is 0 for the undisturbed system) will lead to a spiral point as long as Δ remains negative.

The answer (a) on p. A10 suggests $b = -2$. This changes the matrix to

$$\begin{bmatrix} -2 & -1 \\ -6 & -2 \end{bmatrix}.$$

Hence you now have $p = -4$, $q = 4 - 6 = -2$, so that you obtain a saddle point. Indeed, recall that q is the determinant of the matrix, which is the product of the eigenvalues, and if q is negative, we have two real eigenvalues of opposite signs, as is noted in Table 4.1(b).

To verify all the answers to Prob. 17 given on p. A10, calculate the quantities needed for the perturbed matrix

$$\widetilde{\mathbf{A}} = \begin{bmatrix} b & 1 + b \\ -4 + b & b \end{bmatrix}$$

in the form

$$\begin{aligned} \widetilde{p} &= 2b, \\ \widetilde{q} &= \det \widetilde{\mathbf{A}} = b^2 - (1 + b)(-4 + b) = 3b + 4, \\ \widetilde{\Delta} &= \widetilde{p} - 4\widetilde{q} = 4b^2 - 12b - 16, \end{aligned}$$

and then use Tables 4.1 and 4.2.

Sec. 4.5 Qualitative Methods for Nonlinear Systems

The remarkable basic fact in this section is the following. Critical points of a nonlinear system may be investigated by investigating the critical points of a linear system, obtained by linearization of the given system—a straightforward process of removing nonlinear terms. This is most important because it may be difficult or impossible to solve such a nonlinear system or perhaps even to discuss general properties of solutions.

In the process of linearization, a critical point to be investigated is first moved to the origin of the phase plane and then the nonlinear terms of the transformed system are omitted. This results in a critical point of the same type in almost all cases—exceptions may occur, as is discussed in the text, but this is of lesser importance.

Problem Set 4.5. Page 159

5. Linearization. To determine the critical points of the given system, we set $y_1' = 0$ and $y_2' = 0$, that is,

$$\begin{aligned} y_1' &= y_2 = 0, \\ y_2' &= -y_1 + \tfrac{1}{2}\, y_1^2 = 0. \end{aligned}$$

If we factor the second ODE, that is,

$$y_2' = -y_1\left(1 - \tfrac{1}{2} y_1^2\right) = 0,$$

we get $y_1 = 0$ and $y_2 = 2$. This gives us two critical points of the form (y_1, y_2), that is, $(0, 0)$ and $(2, 0)$. We now discuss one critical point after the other.

The first is at $(0, 0)$, so you need not move it (you do not need to apply a translation). The linearized system is simply obtained by omitting the nonlinear term $\frac{1}{2}y_2^2$. The linearized system is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 \end{aligned} \qquad \text{in vector form} \qquad \mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y}.$$

The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

so that $\lambda_1 = i$, $\lambda_2 = -i$. From this we obtain

$$\begin{aligned} p &= \lambda_1 + \lambda_2 = -i + i = 0, \\ q &= \lambda_1\lambda_2 = (i)(-i) = 1, \\ \Delta &= (\lambda_1 - \lambda_2)^2 = (-2i)^2 = -4. \end{aligned}$$

Since $p = 0$, $q = 1$ and we have pure imaginary eigenvalues, we conclude, by Table 4.1(c) in Sec. 4.4, that we have a center at (0, 0).

Turn to (2, 0). Make a translation such that $(y_1, y_2) = (2, 0)$ becomes $(\widetilde{y}_1, \widetilde{y}_2) = (0, 0)$. *Notation*: Note that the *tilde* over the variables and eigenvalues denotes the *transformed* variables. Nothing needs to be done about y_2, so we set $y_2 = \widetilde{y}_2$. For $y_1 = 2$ we must have $\widetilde{y}_1 = 0$; thus set $y_1 = 2 + \widetilde{y}_1$. This step of translation is always the same in our further work. And if we must translate

$(y_1, y_2) = (a, b)$, we set $y_1 = a + \widetilde{y}_1$, $y_2 = b + \widetilde{y}_2$. The two translations give separate equations, so there is no difficulty.

Now transform the system. The derivatives always simply give $y_1' = \widetilde{y}_1'$, $y_2' = \widetilde{y}_2'$. We thus obtain

$$\begin{aligned}
y_1' &= y_2, \\
y_2' &= -y_1 + \tfrac{1}{2}y_1^2 \\
&= -y_1\left(1 - \tfrac{1}{2}y_1\right) && \text{(by factorization)} \\
&= (-2 - \widetilde{y}_1)\left(1 - \tfrac{1}{2}(2 + \widetilde{y}_1)\right) && \text{(by substitution)} \\
&= (-2 - \widetilde{y}_1)\left(-\tfrac{1}{2}\widetilde{y}_1\right) \\
&= \widetilde{y}_1 - \tfrac{1}{2}\widetilde{y}_1^2 \\
&= \widetilde{y}_2'.
\end{aligned}$$

Thus we have to consider the system (with the second equation obtained by the last two equalities in the above calculation)

$$\begin{aligned}
\widetilde{y}_1' &= \widetilde{y}_2, \\
\widetilde{y}_2' &= \widetilde{y}_1 - \tfrac{1}{2}\widetilde{y}_1^2.
\end{aligned}$$

Hence the system, linearized at the critical (2, 0), is obtained by dropping the term $-\frac{1}{2}\,\widetilde{y}_1^2$, namely,

$$\begin{aligned}
\widetilde{y}_1' &= \widetilde{y}_2 \\
\widetilde{y}_2' &= \widetilde{y}_1
\end{aligned} \qquad \text{in vector form} \qquad \widetilde{\mathbf{y}}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \widetilde{\mathbf{y}}.$$

From this we determine the characteristic equation

$$\det(\widetilde{\mathbf{A}} - \widetilde{\lambda}\mathbf{I}) = \begin{vmatrix} -\widetilde{\lambda} & -1 \\ 1 & -\widetilde{\lambda} \end{vmatrix} = \widetilde{\lambda}^2 - 1 = (\widetilde{\lambda} + 1)(\widetilde{\lambda} - 1) = 0.$$

It has eigenvalues $\lambda = -1$ and $\lambda = 1$. From this we obtain

$$\begin{aligned}
\widetilde{p} &= \widetilde{\lambda}_1 + \widetilde{\lambda}_2 = -1 + 1 = 0, \\
\widetilde{q} &= \widetilde{\lambda}_1\widetilde{\lambda}_2 = (-1)(1) = -1, \\
\widetilde{\Delta} &= (\widetilde{\lambda}_1 - \widetilde{\lambda}_2)^2 = (-1 - 1)^2 = 4.
\end{aligned}$$

Since $\widetilde{q} < 0$ and the eigenvalues are real with opposite sign, we have a saddle point by Table 4.1(b) in Sec. 4.4, which is unstable by Table 4.2(c).

9. Converting nonlinear ODE to a system. Linearization. Critical points. Transform $y'' - 9y + y^3 = 0$ into a system by the usual method (see Theorem 1, p. 135) of setting

$$\begin{aligned}
y_1 &= y, \\
y_2 &= y', \qquad \text{so that} \qquad y_1' = y' = y_2, \text{ and} \\
y_2' &= y'' = 9y - y^3 = 9y_1 - y_1^3.
\end{aligned}$$

Thus the nonlinear ODE, converted to a system of ODEs, is

$$y_1' = y_2,$$
$$y_2' = 9y_1 - y_1^3.$$

To determine the local critical points, we set the right-hand sides of the ODEs in the system of ODEs to 0, that is, $y_1' = y_2 = 0$, $y_2' = 0$. From this, and the second equation, we get

$$y_2' = 9y_1 - y_1^3 = y_1(9 - y_1^2) = y_1(3 + y_1)(3 - y_1) = 0.$$

The critical points are of the form (y_1, y_2). Here we see that $y_2 = 0$ and $y_1 = 0, -3, +3$, so that the critical points are $(0, 0)$, $(-3, 0)$, $(3, 0)$.

Linearize the system of ODEs at $(0, 0)$ by dropping the nonlinear term, obtaining

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 9y_1 \end{aligned} \qquad \text{in vector form} \qquad \mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} \mathbf{y}.$$

From this compute the characteristic polynomial, noting that,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}, \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 9 & -\lambda \end{vmatrix} = \lambda^2 - 9 = (\lambda + 3)(\lambda - 3) = 0.$$

The eigenvalues $\lambda_1 = -3$, $\lambda_2 = 3$. From this we obtain

$$p = \lambda_1 + \lambda_2 = -3 + 3 = 0,$$
$$q = \lambda_1\lambda_2 = (-3)(3) = -9,$$
$$\Delta = (\lambda_1 - \lambda_2)^2 = (-3 - 3)^2 = (-6)^2 = 36.$$

Since $q < 0$ and the eigenvalues are real with opposite signs, we conclude that $(0, 0)$ is a saddle point (by Table 4.1) and, as such, is unstable (Table 4.2).

Turn to $(-3, 0)$. Make a translation such that $(y_1, y_2) = (-3, 0)$ becomes $(\widetilde{y}_1, \widetilde{y}_2) = (0, 0)$. Set $y_1 = -3 + \widetilde{y}_1$, $y_2 = \widetilde{y}_2$. Then

$$\begin{aligned} y_2' &= y_1(9 - y_1^2) \\ &= (-3 + \widetilde{y}_1)\big[9 - (-3 + \widetilde{y}_1)^2\big] \\ &= (-3 + \widetilde{y}_1)\big[9 - (9 - 6\widetilde{y}_1 + \widetilde{y}_1^2)\big] \\ &= (-3 + \widetilde{y}_1)\big[6\widetilde{y}_1 - \widetilde{y}_1^2\big] \\ &= -18\widetilde{y}_1 + 9\widetilde{y}_1^2 - \widetilde{y}_1^3 \\ &= \widetilde{y}_2'. \end{aligned}$$

Thus

$$\widetilde{y}_2' = -18\widetilde{y}_1 + 9\widetilde{y}_1^2 - \widetilde{y}_1^3.$$

Also, by differentiating y_1 in $y_1 = -3 + \widetilde{y}_1$, then using $y_1' = y_2$ from above and then using that $y_2 = \widetilde{y}_2$, you obtain

$$y_1' = \widetilde{y}_1' = y_2 = \widetilde{y}_2.$$

Together we have the transformed system

$$\begin{aligned}\widetilde{y}_1' &= \widetilde{y}_2,\\ \widetilde{y}_2' &= -18\widetilde{y}_1 + 9\widetilde{y}_1^2 - \widetilde{y}_1^3.\end{aligned}$$

To obtain the linearized transformed system, we have to drop the nonlinear terms. They are the quadratic term $9\widetilde{y}_1^2$ and the cubic term $-\widetilde{y}_1^3$. Doing so, we obtain the system

$$\begin{aligned}\widetilde{y}_1' &= \widetilde{y}_2,\\ \widetilde{y}_2' &= -18\widetilde{y}_1.\end{aligned}$$

Expressing it in vector form you have

$$\widetilde{\mathbf{y}}' = \widetilde{\mathbf{A}}\widetilde{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -18 & 0 \end{bmatrix}\begin{bmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \end{bmatrix}.$$

From this we immediately compute the characteristic determinant and obtain the characteristic polynomial, that is,

$$\det(\widetilde{\mathbf{A}} - \widetilde{\lambda}\mathbf{I}) = \begin{vmatrix} -\widetilde{\lambda} & 1 \\ -18 & -\widetilde{\lambda} \end{vmatrix} = \widetilde{\lambda}^2 + 18 = 0.$$

We see that the eigenvalues are complex conjugate, that is, $\widetilde{\lambda}_1 = \sqrt{-18} = (\sqrt{18})(\sqrt{-1}) = (\sqrt{9 \cdot 2})(i) = 3\sqrt{2}i$. Similarly, $\widetilde{\lambda}_2 = -3\sqrt{2}i$. From this, we calculate

$$\begin{aligned}\widetilde{p} &= \widetilde{\lambda}_1 + \widetilde{\lambda}_2 = 3\sqrt{2}i + (-3\sqrt{2}i) = 0,\\ \widetilde{q} &= \widetilde{\lambda}_1\widetilde{\lambda}_2 = 18,\\ \widetilde{\Delta} &= (\widetilde{\lambda}_1 - \widetilde{\lambda}_2)^2 = (6\sqrt{2}i)^2 = -72.\end{aligned}$$

Looking at Tables 4.1, p. 149, and Table 4.2, p. 150, we conclude as follows. Since $\widetilde{p} = 0$, $\widetilde{q} = 18 > 0$, and $\widetilde{\lambda}_1 = 3\sqrt{2}i$, $\widetilde{\lambda}_2 = -3\sqrt{2}i$ are pure imaginary (*see below*), we conclude that $(-3, 0)$ is a center from part (c) of Table 4.1. From Table 4.2(b) we conclude that the critical point $(-3, 0)$ is stable, and indeed a center is stable.

Remark on complex numbers. Complex numbers are of the form $a + bi$, where a, b are real numbers. Now if $a = 0$, so that the complex number is of the form bi, then this complex number is **pure imaginary** (or **purely imaginary**) (i.e, it has no real part a). This is the case with $3\sqrt{2}i$ and $-3\sqrt{2}i$! Thus $6 + 5i$ is not pure imaginary, but $5i$ is pure imaginary.

Similarly the third critical point $(3, 0)$ is a center. If you had trouble with this problem, you may want to do all the calculations for $(3, 0)$ without looking at our calculations for $(-3, 0)$, unless you get very stuck.

13. Nonlinear ODE. We are given a nonlinear ODE $y'' + \sin y = 0$, which we transform into a system of ODEs by the usual method of Sec. 4.1 (Theorem 1) and get

$$\begin{aligned}y_1' &= y_2,\\ y_2' &= -\sin y_1.\end{aligned}$$

Find the location of the critical points by setting the right-hand sides of the two equations in the system to 0, that is, $y_2 = 0$ and $-\sin y_1 = 0$. The sine function is zero at 0, $\pm\pi$, $\pm 2\pi, \ldots$ so that the critical points are at $(\pm n\pi, 0), n = 0, 1, 2$.

Linearize the system of ODEs at $(0, 0)$ approximating

$$\sin y_1 \approx y_1$$

(see Example 1 on p. 153, where this step is justified by a Maclaurin series expansion). This leads to the linearized system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 \end{aligned} \qquad \text{in vector form} \qquad \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y}.$$

Using (5) in Sec. 4.4 we have

$$\begin{aligned} p &= a_{11} + a_{22} = 0 + 0 = 0, \\ q &= \det \mathbf{A} = 0 \cdot 0 - (-1)(-1) = 1, \\ \Delta &= p^2 - 4q = 0^2 - 4 \cdot 1 = -4. \end{aligned}$$

From this, and Table 4.1(c) in Sec. 4.4, we conclude that $(0, 0)$ is a center. As such it is always stable, as Table 4.2(b) confirms. Since the sine function has a period of 2π, we conclude that $(0, 0), (\pm 2\pi, 0), (\pm 4\pi, 0), \ldots$ are centers.

Consider $(\pi, 0)$. We transform the critical points to $(0, 0)$ as explained at the beginning of Sec. 4.5. This is done by the translation $y_1 = \pi + \widetilde{y}_1$, $y_2 = \widetilde{y}_2$. We now have to determine the transformed system:

$$\begin{aligned} \widetilde{y}_1 &= y_1 - \pi \qquad \text{so} \qquad \widetilde{y}_1' = y_1' = y_2 \qquad \text{so that} \qquad \widetilde{y}_1' = \widetilde{y}_2 \\ \widetilde{y}_2 &= y_2 \qquad \text{so} \qquad \widetilde{y}_2' = y_2' = -\sin y = -\sin(\pi + \widetilde{y}_1). \end{aligned}$$

Now

$$\begin{aligned} -\sin(\pi + \widetilde{y}_1) &= \sin(-(\pi + \widetilde{y}_1)) \qquad \text{(since sine is an odd function, App. 3, Sec. A3.1)} \\ &= \sin(-\widetilde{y}_1 - \pi) = -\sin \widetilde{y}_1 \cdot \cos\pi - \cos\widetilde{y}_1 \cdot \sin\pi \quad \text{[by (6), Sec. A3.1 in App. 3]} \\ &= \sin\widetilde{y}_1 \qquad (\text{since } \cos\pi = -1, \sin\pi = 0). \end{aligned}$$

Hence

$$\begin{aligned} \widetilde{y}_1' &= \widetilde{y}_2, \\ \widetilde{y}_2' &= \sin\widetilde{y}_1. \end{aligned}$$

Linearization gives the system

$$\begin{aligned} \widetilde{y}_1' &= \widetilde{y}_2 \\ \widetilde{y}_2' &= \widetilde{y}_1 \end{aligned} \qquad \text{in vector form} \qquad \widetilde{\mathbf{y}}' = \widetilde{\mathbf{A}}\widetilde{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \widetilde{\mathbf{y}}.$$

We compute

$$\begin{aligned} \widetilde{p} &= \widetilde{a}_{11} + \widetilde{a}_{22} = 0 + 0 = 0, \\ \widetilde{q} &= \det \widetilde{\mathbf{A}} = 0 \cdot 0 - 1 \cdot 1 = -1, \\ \Delta &= p^2 - 4q = 0^2 - 4 \cdot (-1) = 4. \end{aligned}$$

Since $\widetilde{q} < 0$, Table 4.1(b) shows us that we have a saddle point. By periodicity, $(\pm 3\pi, 0), (\pm 5\pi, 0)$, $(\pm 7\pi, 0) \ldots$ are saddle points.

Sec. 4.6 Nonhomogeneous Linear Systems of ODEs

In this section we return from nonlinear to linear systems of ODEs. The text explains that the transition from homogeneous to nonhomogeneous linear systems is quite similar to that for a single ODE. Namely, since a general solution is the sum of a general solution $\mathbf{y}^{(h)}$ of the homogeneous system plus a particular solution $\mathbf{y}^{(p)}$ of the nonhomogeneous system, your main task is the determination of a $\mathbf{y}^{(p)}$, either by undetermined coefficients or by variation of parameters. Undetermined coefficients is explained on p. 161. It is similar to that for single ODEs. The only difference is that in the Modification Rule you may need an extra term. For instance, if e^{kt} appears in $\mathbf{y}^{(h)}$, set $\mathbf{y}^{(p)} = \mathbf{u}te^{kt} + \mathbf{v}e^{kt}$ with the extra term $\mathbf{v}e^{kt}$.

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3. General solution. e^{3t} and $-3e^{3t}$ are such that we can apply the method of undetermined coefficients for determining a particular solution of the nonhomogeneous system. For this purpose we must first determine a general solution of the homogeneous system. The matrix of the latter is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It has the characteristic equation $\lambda^2 - 1 = 0$. Hence the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1$ and $\lambda_2 = 1$. Eigenvectors $\mathbf{x} = \mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are obtained from $(\mathbf{A} - \lambda I)\mathbf{x} = 0$ with $\lambda = \lambda_1 = -1$ and $\lambda = \lambda_2 = 1$, respectively. For $\lambda_1 = -1$ we obtain

$$x_1 + x_2 = 0, \quad \text{thus} \quad x_2 = -x_1, \quad \text{say,} \quad x_1 = 1, x_2 = -1.$$

Similarly, for $\lambda_2 = 1$ we obtain

$$-x_1 + x_2 = 0, \quad \text{thus} \quad x_2 = x_1, \quad \text{say,} \quad x_1 = 1, x_2 = 1.$$

Hence eigenvectors are $\mathbf{x}^{(1)} = [1 \quad -1]^\mathsf{T}$ and $\mathbf{x}^{(2)} = [1 \quad 1]^\mathsf{T}$. This gives the general solution of the homogeneous system

$$\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}.$$

Now determine a particular solution of the nonhomogeneous system. Using the notation in the text (Sec. 4.6) we have on the right $\mathbf{g} = [1 \quad -3]^\mathsf{T} e^{3t}$. This suggests the choice

$$\text{(a)} \qquad \mathbf{y}^{(p)} = \mathbf{u}e^{3t} = [u_1 \quad u_2]^\mathsf{T} e^{3t}.$$

Here $\mathbf{u}$ is a constant vector to be determined. The Modification Rule is not needed because 3 is not an eigenvalue of $\mathbf{A}$. Substitution of (a) into the given system $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$ yields

$$\mathbf{y}^{(p)\prime} = 3\mathbf{u}e^{3t} = \mathbf{A}\mathbf{y}^{(p)} + \mathbf{g} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{3t}$$

Omitting the common factor e^{3t}, you obtain, in terms of components,

$$\begin{aligned} 3u_1 &= u_2 + 1 \qquad \text{ordered} \qquad & 3u_1 - u_2 &= 1, \\ 3u_2 &= u_1 - 3 & -u_1 + 3u_2 &= -3. \end{aligned}$$

Solution by elimination or by Cramer's rule (Sec. 7.6) $u_1 = 0$ and $u_2 = -1$. Hence the answer is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{3t}.$$

13. Initial value problem. The given system is

$$\begin{aligned} y_1' &= \quad y_2 - \ 5 \sin t, \\ y_2' &= -4y_1 + 17 \cos t, \end{aligned}$$

where the initial conditions are $y_1(0) = 5, y_2(0) = 2$. First we have to solve the homogeneous system

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}.$$

Its characteristic equation $\lambda^2 + 4 = 0$ has the roots $\pm 2\,i$. For $\lambda_1 = 2\,i$ obtain an eigenvector from $-2\,i\,x_1 + x_2 = 0$, say, $\mathbf{x}^{(1)} = [1 \quad 2i]^{\mathsf{T}}$. For $\lambda = -2\,i$ we have $2\,i\,x_1 + x_2 = 0$, so that an eigenvector is, say, $\mathbf{x}^{(2)} = [1 \quad -2i]^{\mathsf{T}}$. You obtain the complex general solution of the homogeneous system as follows. We apply Euler's formula twice.

$$\begin{aligned} \mathbf{y}^{(h)} &= c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it} \\ &= c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} (\cos 2t + i \sin 2t) + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} (\cos(-2t) + i \sin(-2t)) \\ &= c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} (\cos 2t + i \sin 2t) + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} (\cos 2t + i \sin 2t) \\ &= \begin{bmatrix} c_1 \cos 2t + \ \ ic_1 \sin 2t + \ \ c_2 \cos 2t - \ \ ic_2 \sin 2t \\ 2ic_1 \cos 2t + 2i^2 c_1 \sin 2t - 2ic_2 \cos 2t + 2i^2 c_2 \sin 2t \end{bmatrix} \\ &= \begin{bmatrix} c_1 \cos 2t + ic_1 \sin 2t + \ \ c_2 \cos 2t - ic_2 \sin 2t \\ 2ic_1 \cos 2t - 2c_1 \sin 2t - 2ic_2 \cos 2t - 2c_2 \sin 2t \end{bmatrix} \\ &= \begin{bmatrix} (c_1 + c_2) \cos 2t + \quad i(c_1 - c_2) \sin 2t \\ (2ic_1 - 2ic_2) \cos 2t + (-2c_1 - 2c_2) \sin 2t \end{bmatrix} \\ &= \begin{bmatrix} c_1 + c_2 \\ 2ic_1 - 2ic_2 \end{bmatrix} \cos 2t + i \begin{bmatrix} c_1 - c_2 \\ 2ic_1 + 2ic_2 \end{bmatrix} \sin 2t \\ &= \begin{bmatrix} A \\ B \end{bmatrix} \cos 2t + \begin{bmatrix} \frac{1}{2} B \\ -2A \end{bmatrix} \sin 2t \end{aligned}$$

where

$$A = c_1 + c_2, \qquad B = 2i\,(c_1 - c_2).$$

We determine $\mathbf{y}^{(p)}$ by the method of undetermined coefficients, starting from

$$\mathbf{y}^{(p)} = \mathbf{u}\cos t + \mathbf{v}\sin t = \begin{bmatrix} u_1 \cot t + v_1 \sin t \\ u_2 \cos t + v_2 \sin t \end{bmatrix}.$$

Termwise differentiation yields

$$\mathbf{y}^{(p)\prime} = \begin{bmatrix} -u_1 \sin t + v_1 \sin t \\ -u_2 \sin t + v_2 \cos t \end{bmatrix}.$$

In components

$$\begin{aligned} y_1^{(p)\prime} &= -u_1 \sin t + v_1 \sin t, \\ y_2^{(p)\prime} &= -u_2 \sin t + v_2 \cos t. \end{aligned}$$

Substituting this and its derivative into the given nonhomogeneous system, we obtain, in terms of components,

$$\begin{aligned} -u_1 \sin t + v_1 \sin t &= u_2 \cos t + v_2 \sin t - 5 \sin t, \\ -u_2 \sin t + v_2 \cos t &= -4u_1 \cos t - 4v_1 \sin t + 17 \cos t. \end{aligned}$$

By equating the coefficients of the cosine and sine in the first of these two equations, we obtain

$$\begin{aligned} &(E1) \quad -u_1 = v_2 - 5, && (E2) \quad v_1 = u_2, \\ &(E3) \quad -u_2 = -4v_2, && (E4) \quad v_2 = -4u_1 + 17. \end{aligned}$$

Substituting $(E4)$ into $(E1)$ $\quad -u_1 = -4u_1 + 17 - 5 \quad$ gives $(E5)$ $u_1 = \frac{12}{3} = 4$.

Substituting $(E5)$ into $(E4)$ $\quad v_2 = -4(4) + 17 = 1 \quad$ gives $(E6)$ $v_2 = 1$.

$(E2)$ and $(E3)$ together $\quad u_2 = v_1 \quad$ and $\quad u_2 = 4v_1$ is only true for $(E7)$ $u_2 = v_1 = 0$.

Equations $(E5)$, $(E6)$, $(E7)$ form the solution to the homogeneous linear system, that is,

$$u_1 = 4, \quad u_2 = 0, \quad v_1 = 0, \quad v_2 = 1.$$

This gives the general answer

$$\begin{aligned} y_1 &= y_1^{(h)} + y_1^{(p)} = A\cos 2t + \tfrac{1}{2}B \sin 2t + 4\cos t, \\ y_2 &= y_2^{(h)} + y_2^{(p)} = B\cos 2t + \tfrac{1}{2}A \sin 2t + \sin t. \end{aligned}$$

To solve the initial value problem, we use $y_1(0) = 5, y_2(0) = 2$ to obtain

$$\begin{aligned} y_1(0) &= A\cos 0 + \tfrac{1}{2}B\sin 0 + 4\cos 0 = A \cdot 1 + 0 + 4 = 5 \quad \text{hence} \quad A = 1, \\ y_2(0) &= B\cos 0 - 2A\sin 0 + \sin 0 = B \cdot 1 - 0 + 0 = 2 \quad \text{hence} \quad B = 2. \end{aligned}$$

Thus the final answer is

$$y_1 = \cos 2t + \sin 2t + 4\cos t,$$
$$y_2 = 2\cos 2t - 2\sin 2t + \sin t.$$

17. Network. First derive the model. For the left loop of the electrical network you obtain, from Kirchhoff's Voltage Law

(a) $$LI_1' + R_1(I_1 - I_2) = E$$

because both currents flow through R_1, but in opposite directions, so that you have to take their difference. For the right loop you similarly obtain

(b) $$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2\,dt = 0.$$

Insert the given numerical values in (a). Do the same in (b) and differentiate (b) in order to get rid of the integral. This gives

$$I_1' + 2(I_1 - I_2) = 200,$$
$$2(I_2' - I_1') + 8I_2' + 2I_2 = 0.$$

Write the terms in the first of these two equations in the usual order, obtaining

(a1) $$I_1' = -2I_1 + 2I_2 + 200.$$

Do the same in the second equation as follows. Collecting terms and then dividing by 10, you first have

$$10I_2' - 2I_1' + 2I_2 = 0 \qquad \text{and then} \qquad I_2' - 0.2I_1' + 0.2I_2 = 0.$$

To obtain the usual form, you have to get rid of the term in I_1', which you replace by using (a1). This gives

$$I_2' - 0.2(-2I_1 + 2I_2 + 200) + 0.2I_2 = 0.$$

Collecting terms and ordering them as usual, you obtain

(b1) $$I_2' = -0.4I_1 + 0.2I_2 + 40.$$

(a1) and (b1) are the two equations of the system that you use in your further work. The matrix of the corresponding homogeneous system is

$$\mathbf{A} = \begin{bmatrix} -2 & 2 \\ -0.4 & 0.2 \end{bmatrix}.$$

Its characteristic equation is (**I** is the unit matrix)

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-2 - \lambda)(0.2 - \lambda) - (-0.4)\cdot 2 = \lambda^2 + 1.8\lambda + 0.4 = 0.$$

This gives the eigenvalues

$$\lambda_1 = -0.9 + \sqrt{0.41} = -0.259688$$

and

$$\lambda_2 = -0.9 - \sqrt{0.41} = -1.540312.$$

Eigenvectors are obtained from $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with $\lambda = \lambda_1$ and $\lambda = \lambda_2$. For λ_1 this gives

$$(-2 - \lambda_1)x_1 + 2x_2 = 0, \quad \text{say,} \quad x_1 = 2 \quad \text{and} \quad x_2 = 2 + \lambda_1.$$

Similarly, for λ_2 you obtain

$$(-2 - \lambda_2)x_1 + 2x_2 = 0, \quad \text{say,} \quad x_1 = 2 \quad \text{and} \quad x_2 = 2 + \lambda_2.$$

The eigenvectors thus obtained are

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 2 + \lambda_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1.1 + \sqrt{0.41} \end{bmatrix},$$

and

$$\mathbf{x}^{(2)} = \begin{bmatrix} 2 \\ 2 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1.1 - \sqrt{0.41} \end{bmatrix}.$$

This gives as a general solution of the homogeneous system

$$\mathbf{I}^{(h)} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t}.$$

You finally need a particular solution $\mathbf{I}^{(p)}$ of the given nonhomogeneous system $\mathbf{J}' = \mathbf{AJ} + \mathbf{g}$, where $\mathbf{g} = [200 \quad 40]^\mathsf{T}$ is constant, and $\mathbf{J} = [I_1 \quad I_2]^\mathsf{T}$ is the vector of the currents. The method of undetermined coefficients applies. Since $\mathbf{g}$ is constant, you can choose a constant $\mathbf{I}^{(p)} = \mathbf{u} = [u_1 \quad u_2]^\mathsf{T} = \mathbf{const}$ and substitute it into the system, obtaining, since $\mathbf{u}' = \mathbf{0}$,

$$\mathbf{I}^{(p)\prime} = \mathbf{0} = \mathbf{Au} + \mathbf{g} = \begin{bmatrix} -2 & 2 \\ -0.4 & 0.2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 200 \\ 40 \end{bmatrix} = \begin{bmatrix} -2u_1 + 2u_2 + 200 \\ -0.4u_1 + 0.2u_2 + 40 \end{bmatrix}.$$

Hence you can determine u_1 and u_2 from the system

$$\begin{aligned} -2u_1 + 2u_2 \quad &= -200, \\ -0.4u_1 + 0.2u_2 &= -40. \end{aligned}$$

The solution is $u_1 = 100$, $u_2 = 0$. The answer is

$$\mathbf{J} = \mathbf{I}^{(h)} + \mathbf{I}^{(p)}.$$

Chap. 5 Series Solutions of ODEs. Special Functions

We continue our studies of ODEs with *Legendre's, Bessel's,* and the *hypergeometric* equations. These ODEs have *variable* coefficients (in contrast to previous ODEs with *constant* coefficients) and are of great importance in applications of physics and engineering. Their solutions require the use of special functions, which are functions that do not appear in elementary calculus. Two very important classes of special functions are the **Legendre polynomials** (Sec. 5.2) and the **Bessel functions** (Secs. 5.4, 5.5). Although these, and the many other special functions of practical interest, have quite different properties and serve very distinct purposes, it is most remarkable that these functions are accessible by the same mathematical construct, namely, **power series**, perhaps multiplied by a fractional power or a logarithm.

As an engineer, applied mathematician, or physicist you have to know about special functions. They not only appear in ODEs but also in PDEs and numerics. Granted that your CAS knows all the functions you will ever need, you still need a road map of this topic as provided by Chap. 5 to be able to navigate through the wide field of relationships and formulas and to be able to select what functions and relations you need for your particular engineering problem. Furthermore, getting a good understanding of this material will aid you in finding your way through the vast literature on special functions and its applications. Such research may be necessary for solving particular engineering problems.

Sec. 5.1 Power Series Method

Section 5.1 is probably, to some extent, familiar to you, as you have seen some simple examples of power series (2) in calculus. **Example 2**, pp. 168–169 of the text, explains the power series method for solving a simple ODE. Note that we always start with (2) and differentiate (2), once for first-order ODEs and twice for second-order ODEs, etc., as determined by the order of the given ODE. Furthermore, be aware that you may be able to simplify your final answer (as was done in this example) by being able to recognize what function is represented by the power series. This requires that you know important power series for functions as say, given in **Example 1**. **Example 3** shows a special Legendre equation and foreshadows the things to come.

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5. Radius of convergence. The radius of convergence of a power series is an important concept. A power series in powers of x may converge for all x (this is the best possible case), within an interval with the center x_0 as midpoint (in the complex plane: within a disk with center z_0), or only at the center (the practically useless case). In the second case, the interval of convergence has length $2R$, where R is called the *radius of convergence* (it is a radius in the complex case, as has just been said) and is given by (11a) or (11b) on p. 172. Here it is assumed that the limits in these formulas exist. This will be the case in most applications. (For help when this is not the case, see Sec. 15.2.) The convergence radius is important whenever you want to use series for computing values, exploring properties of functions represented by series, or proving relations between functions, tasks of which you will gain a first impression in Secs. 5.2–5.5 and corresponding problems.

We are given the power series

$$\sum_{m=0}^{\infty}\left(\frac{2}{3}\right)^m x^{2m} = 1 + \frac{2}{3}x^2 + \frac{4}{9}x^4 + \frac{8}{27}x^6 + \cdots .$$

The series is in powers of $t = x^2$ with coefficients $a_m = \left(\frac{2}{3}\right)^m$, so that in (11b)

$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{\left(\frac{2}{3}\right)^{m+1}}{\left(\frac{2}{3}\right)^m}\right| = \frac{2}{3}.$$

Thus

$$R = \frac{1}{\lim\limits_{m\to\infty} \left(\frac{2}{3}\right)} = \frac{3}{2}.$$

Hence, the series converges for $|t| = |x^2| < \frac{3}{2}$, that is, $|x| < \sqrt{\frac{3}{2}}$. The radius of convergence of the given series is thus $\sqrt{\frac{3}{2}}$. Try using (11a) to see that you get the same result.

9. Power series method. The ODE $y'' + y = 0$ can be solved by the method of Sec. 2.2, first computing the characteristic equation. However, for demonstrating the power series method, we proceed as follows.

Step 1. Compute y, y', and y'' using power series.

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots = \sum_{m=0}^{\infty} a_m x^m.$$

Termwise differentiating the series for y gives a series for y':

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + 7a_7x^6 + \cdots = \sum_{m=0}^{\infty} m a_m x^{m-1}.$$

Differentiating again

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots = \sum_{m=0}^{\infty} m(m-1)a_m x^{m-2}.$$

Step 2. Insert the power series obtained for y and y'' into the given ODE and align the terms vertically by powers of x:

$$\begin{array}{llllll} 2a_2 + & 3\cdot 2a_3x + & 4\cdot 3a_4x^2 + & 5\cdot 4a_5x^3 + & 6\cdot 5a_6x^4 + & 7\cdot 6a_7x^5 + \cdots \\ a_0 + & a_1x + & a_2x^2 + & a_3x^3 + & a_4x^4 + & a_5x^5 + \cdots = 0. \end{array}$$

Step 3. Add like powers of x. Equate the sum of the coefficients of each occurring power of x to 0:

$$\begin{array}{llll} [0] & [x^0] & 2a_2 + a_0 = 0 & a_2 = -\frac{1}{2}a_0 \qquad\qquad\qquad\quad = -\frac{1}{2!}a_0 \\[2ex] [1] & [x^1] & 6a_3 + a_1 = 0 & a_3 = -\frac{1}{6}a_1 \qquad\qquad\qquad\quad = -\frac{1}{3!}a_1 \\[2ex] [2] & [x^2] & 12a_4 + a_2 = 0 & a_4 = -\frac{1}{12}a_2 = -\frac{1}{12}\left(-\frac{1}{2!}a_0\right) = \frac{1}{4!}a_0 \\[2ex] [3] & [x^3] & 20a_5 + a_3 = 0 & a_5 = -\frac{1}{20}a_3 = -\frac{1}{20}\left(-\frac{1}{3!}a_1\right) = \frac{1}{5!}a_1 \\[2ex] [4] & [x^4] & 30a_6 + a_4 = 0 & a_6 = -\frac{1}{30}a_4 = -\frac{1}{30}\cdot\frac{1}{4!}a_0 \quad = -\frac{1}{6!}a_0 \\[2ex] [5] & [x^5] & 42a_7 + a_5 = 0 & a_7 = -\frac{1}{42}a_5 = -\frac{1}{42}\cdot\frac{1}{5!}a_1 \quad = -\frac{1}{7!}a_1 \end{array}$$

Step 4. Write out the solution to the ODE by substituting the values for $a_2, a_3, a_4, \ldots$ computed in Step 3.

We obtain

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x + \cdots \\ &= a_0 + a_1x - \frac{1}{2!}a_0x^2 - \frac{1}{3!}a_1x^3 + \frac{1}{4!}a_0x^4 + \frac{1}{5!}a_1x^5 - \frac{1}{6!}a_0x^6 - \frac{1}{7!}a_1x^7 + + - - \cdots \\ &= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + - \cdots\right) + a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + - \cdots\right) \end{aligned}$$

Step 5. Identify what functions are represented by the series obtained. We find

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + - \cdots = \sum_{m=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = \cos x$$

and

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + - \cdots = \sum_{m=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sin x.$$

Substituting these into the series obtained in Step 4, we see that

$$\begin{aligned} y &= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + - \cdots\right) + a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + - \cdots\right) \\ &= a_0 \cos x + a_1 \sin x. \end{aligned}$$

15. Shifting summation indices. For the first sum

$$\sum_{s=2}^{\infty} \frac{s(s+1)}{s^2+1}x^{s-1}.$$

To make the power under the summation sign be x^m, we have to set

$$s - 1 = m \quad \text{so that } s = m + 1.$$

Then, by substituting $m + 1$ for s, we obtain

$$\frac{s(s+1)}{s^2+1}x^{s-1} = \frac{(m+1)(m+2)}{(m+1)^2+1}x^m$$

and the summation goes from $m = s - 1 = 2 - 1 = 1$ (since $s = 2$) to ∞. If we put it all together

$$\sum_{s=2}^{\infty} \frac{s(s+1)}{s^2+1}x^{s-1} = \sum_{m=1}^{\infty} \frac{(m+1)(m+2)}{(m+1)^2+1}x^m.$$

(Write out a few terms of each series to verify the result.) Similarly for the second sum

$$\sum_{p=1}^{\infty} \frac{p^2}{(p+1)!}x^{p+4}$$

we set

$$p + 4 = m \quad \text{so that } p = m - 4.$$

Hence

$$\frac{p^2}{(p+1)!}x^{p+4} = \frac{(m-4)^2}{(m-4+1)!}x^m = \frac{(m-4)^2}{(m-3)!}x^m.$$

This gives the final answer

$$\sum_{p=1}^{\infty} \frac{p^2}{(p+1)!}x^{p+4} = \sum_{m=5}^{\infty} \frac{(m-4)^2}{(m-3)!}x^m.$$

Sec. 5.2 Legendre's Equation. Legendre Polynomials $\mathbf{P}_n(x)$

Note well that Legendre's equation involves the parameter n, so that (1) is actually a whole family of ODEs, with basically different properties for different n. In particular, for integer $n = 0, 1, 2, \cdots$, one of the series (6) or (7) reduces to a polynomial, and it is remarkable that these "simplest" cases are of particular interest in applications. Take a look at **Fig. 107**, on p. 178. It graphs the Legendre polynomials P_0, P_1, P_2, P_3, and P_4.

Problem Set 5.2. Page 179

1. Legendre functions for $n = 0$. The power series and Frobenius methods were instrumental in establishing large portions of the very extensive theory of special functions (see, for instance, Refs. [GenRef1], [GenRef10] in Appendix 1 of the text), as needed in engineering, physics (astronomy!), and other areas. This occurred simply because many special functions appeared first in the form of power series solutions of differential equations. In general, this concerns properties and relationships between higher transcendental functions. The point of Prob. 1 is to illustrate that sometimes such functions may reduce to elementary functions known from calculus. If we set $n = 0$ in (7), we observe that $y_2(x)$ becomes $\frac{1}{2}\ln((1+x)/(1-x))$. In this case, the answer suggests using

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - + \cdots.$$

Replacing x by $-x$ and multiplying by -1 on both sides gives

$$\ln\frac{1}{1-x} = -\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots.$$

Addition of these two series and division by 2 verifies the last equality sign in the formula of Prob. 1. We are requested to obtain this result directly by solving the Legendre equation (1) with $n = 0$, that is,

$$(1-x^2)y'' - 2xy' = 0 \quad \text{or} \quad (1-x^2)z' = 2xz, \quad \text{where } z = y'.$$

Separation of variables and integration gives

$$\frac{dz}{z} = \frac{2x}{1-x^2}dx, \quad \ln|z| = -\ln|1-x^2| + c, \quad z = \frac{C_1}{1-x^2}.$$

y is now obtained by another integration, using partial fractions:

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x-1}\right).$$

This gives

$$y = \int z\,dx = \frac{1}{2}C_1(\ln(x+1) - \ln(x-1)) + c = \frac{1}{2}C_1 \ln\frac{x+1}{x-1} + c.$$

Since $y_1(x)$ in (6), Sec. 5.2, p. 176, reduces to 1 if $n = 0$, we can now readily express the solution we obtained in terms of the standard functions y_1 and y_2 in (6) and (7), namely,

$$y = cy_1(x) + C_1 y_2(x).$$

11. **ODE.** Set $x = az$, thus $z = x/a$, and apply the chain rule, according to which

$$\frac{d}{dx} = \frac{d}{dz}\frac{dz}{dx} = \frac{1}{a}\frac{d}{dz} \quad \text{and} \quad \frac{d^2}{dx^2} = \frac{1}{a^2}\frac{d^2}{dz^2}.$$

Substitution now gives

$$(a^2 - a^2z^2)\frac{d^2y}{dz^2}\frac{1}{a^2} - 2az\frac{dy}{dz}\frac{1}{a} + n(n+1)y = 0.$$

The a factors cancel and you are left with

$$(1-z^2)y'' - 2zy' + n(n+1)y = 0.$$

Hence two independent solutions are $P_n(z) = P_n(x/a)$ and $Q_n(x/a)$, so that the general solution is any linear combination of these as claimed in Appendix 2, p. A12.

Sec. 5.3 Extended Power Series Method: Frobenius Method

The first step in using the **Frobenius method** is to see whether the given ODE is in standard form (1) on p. 180. If not, you may have to divide it by, say x^2 (as in our next example) or otherwise. Once the ODE is in standard form, you readily determine $b(x)$ and $c(x)$. The constant terms of $b(x)$ and $c(x)$ are b_0 and c_0, respectively. Once you have determined them, you can set up the *indicial equation* (4) on p. 182. Next you determine the roots of the indicial equation. You have three cases: **Case 1** (*distinct roots not differing by an integer*), **Case 2** (*a double root*), and **Case 3** (*roots differing by an integer*). The type of case determines the solution as discussed in **Theorem 2**, pp. 182–183.

For instance, a typical ODE that can be solved by the Frobenius method is

$$x^2y'' + 4xy' + (x^2+2)y = 0.$$

Dividing by x^2 to get it into the form (1), as required by Theorem 1 (Frobenius method), you have

$$y'' + \frac{4}{x}y' + \frac{x^2+2}{x^2}y = 0.$$

You see that $b(x) = 4$ and $c(x) = x^2 + 2$. Hence you have $b_0 = 4$, $c_0 = 2$, so that the indicial equation is

$$r(r-1) + 4r + 2 = r^2 + 3r + 2 = (r+2)(r+1) = 0,$$

and the roots are -2 and -1. The roots differ by the integer 1. Thus Case 3 ("Roots differing by an integer") of Theorem 2, pp. 182–183, applies.

An outline on how to solve the **hypergeometric equation** (15) is given in **Team Project 14** of the problem set. Typical ODEs of type (15) are given in **Problems 15–20**.

Problem Set 5.3. Page 186

3. Basis of solutions by the Frobenius method. Case 3: Roots differ by integer. Substitute y, y', and y'', given by (2), p. 180 and (2*), p. 181 into the differential equation $xy'' + 2y' + xy = 0$. This gives

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} + \sum_{m=0}^{\infty} 2(m+r)a_m x^{m+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0.$$

The first two series have the same general power, and you can take them together. In the third series set $n = m - 2$ to get the same general power. $n = 0$ then gives $m = 2$. You obtain

$$\text{(A)} \qquad \sum_{m=0}^{\infty}(m+r)(m+r+1)a_m x^{m+r-1} + \sum_{m=2}^{\infty} a_{m-2} x^{m+r-1} = 0.$$

For $m = 0$ this gives the indicial equation

$$r(r+1) = 0.$$

The roots are $r = 0$ and -1. They differ by an integer. This is Case 3 of Theorem 2 on p. 183. Consider the larger root $r = 0$. Then (A) takes the form

$$\sum_{m=0}^{\infty} m(m+1)a_m x^{m-1} + \sum_{m=2}^{\infty} a_{m-2} x^{m-1} = 0.$$

$m = 1$ gives $2a_1 = 0$. Because the subscripts on the a's differ by 2, this implies $a_3 = a_5 = \cdots = 0$, as is seen by taking $m = 3, 5, \cdots$. Furthermore,

$$m = 2 \text{ gives } 2 \cdot 3a_2 + a_0 = 0, \text{ hence } a_0 \text{ arbitrary, } a_2 = -\frac{a_0}{3!}$$
$$m = 4 \text{ gives } 4 \cdot 5a_4 + a_2 = 0, \text{ hence } a_4 = -\frac{a_2}{4 \cdot 5} = +\frac{a_0}{5!}$$

and so on. Since you want a basis and a_0 is arbitrary, you can take $a_0 = 1$. Recognize that you then have the Maclaurin series of

$$y_1 = \frac{\sin x}{x}.$$

Now determine an independent solution y_2. Since, in Case 3, one would have to assume a term involving the logarithm (which may turn out to be zero), reduction of order (Sec. 2.1) seems to be simpler. This begins by writing the equation in standard form (divide by x):

$$y'' + \left(\frac{2}{x}\right)y' + y = 0.$$

In (2) of Sec. 2.1 you then have $p = 2/x$, $-\int p\,dx = -2\ln|x| = \ln(1/x^2)$, hence $\exp(-\int p\,dx) = 1/x^2$. Insertion of this and y_1^2 into (9) and cancellation of a factor x^2 gives

$$U = \frac{1}{\sin^2 x}, \quad u = \int U\,dx = -\cot x, \quad y_2 = uy_1 = -\frac{\cos x}{x}.$$

From this you see that the general solution is a linear combination of the two independent solutions, that is, $c_1y_1 + c_2y_2 = c_1(\sin x/x) + c_2(-(\cos x/x))$.

In particular, then you know that, since $-\cos x/x$ is a solution, then so is $c_2(-\cos x/x)$ for $c_2 = -1$. This means that you can, for beautification, get rid of the minus sign in that way and obtain $\cos x/x$ as the second independent solution. This then corresponds exactly to the answer given on p. A12.

13. **Frobenius method. Case 2: Double root.** To determine r_1, r_2 from (4), we multiply both sides of the given ODE $xy'' + (1-2x)y' + (x-1)y = 0$ by x and get it in the form (1′)

$$x^2y'' + (x - 2x^2)y' + (x^2 - x)y = 0.$$

The ODE is of the form (see text p. 181)

$$x^2y'' + xb(x)y' + c(x)y = 0.$$

Hence

$$xb(x) = x - 2x^2 = x(1-2x), \quad b(x) = 1 - 2x.$$
$$c(x) = x^2 - x.$$

Also

$$b_0 = 1, \quad c_0 = 0 \ \text{ (no constant term in } c(x)).$$

Thus the indical equation of the given ODE is

$$r(r-1) + b_0r + c_0 = r(r-1) + 1\cdot r + 0 = 0, \ \ r^2 - r + r = 0, \ \ \text{so that } r^2 = 0.$$

Hence the indical equation has the double root $r = 0$.

First solution. We obtain the first solution by substituting (2) with $r = 0$ into the given ODE (in its original form).

$$y = \sum_{m=0}^{\infty} a_m x^{m+r} = \sum_{m=0}^{\infty} a_m x^m$$
$$y' = \sum_{m=0}^{\infty} m a_m x^{m-1}$$
$$y'' = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.$$

Substitution into the ODE gives

$$\begin{aligned}
xy'' + (1-2x)y' + (x-1)y &= x\sum_{m=0}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=0}^{\infty} m a_m x^{m-1} - 2x\sum_{m=0}^{\infty} m a_m x^{m-1} \\
&\quad + x\sum_{m=0}^{\infty} a_m x^m - \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=0}^{\infty} m(m-1)a_m x^{m-1} + \sum_{m=0}^{\infty} m a_m x^{m-1} - 2\sum_{m=0}^{\infty} m a_m x^m \\
&\quad + \sum_{m=0}^{\infty} a_m x^{m+1} - \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=0}^{\infty} [m(m-1)a_m x^{m-1} + m a_m x^{m-1} - 2m a_m x^m + a_m x^{m+1} - a_m x^m] \\
&= \sum_{m=0}^{\infty} m^2 a_m x^{m-1} - \sum_{m=0}^{\infty} [2m+1]a_m x^m + \sum_{m=0}^{\infty} a_m x^{m+1} \\
&= \sum_{s=-1}^{\infty} (s+1)^2 a_{s+1} x^s - \sum_{s=0}^{\infty} [2s+1]a_s x^s + \sum_{s=1}^{\infty} a_{s-1} x^s = 0.
\end{aligned}$$

Now, for $s = -1$,

$$(0)^2 a_0 = 0,$$

for $s = 0$,

$$a_1 - a_0 = 0,$$

and for $s > 0$,

$$\text{(A)} \qquad (s+1)^2 a_{s+1} - (2s+1)a_s + a_{s-1} = 0.$$

For $s = 1$ we obtain

$$(2)^2 a_2 - (2+1)a_1 + a_0 = 0, \quad 4a_2 - 3a_1 + a_0 = 0.$$

Substituting $a_1 = a_0$ into the last equation

$$4a_2 - 3a_0 + a_0 = 0, \quad 4a_2 = 2a_0, \quad a_2 = \tfrac{1}{2}a_0.$$

We could solve (A) for a_{s+1}

$$\begin{aligned}
(s+1)^2 a_{s+1} &= (2s+1)a_s + a_{s-1} \\
a_{s+1} &= \frac{1}{(s+1)^2}[(2s+1)a_s - a_{s-1}].
\end{aligned}$$

For $s = 2$, taking $a_0 = 1$, this gives

$$a_3 = \frac{1}{3^2}[5a_2 - a_1] = \frac{1}{3^2}\left[5 \cdot \frac{1}{2} - 1\right] = \frac{1}{3^2}\left[\frac{3}{2}\right] = \frac{1 \cdot 3}{3^2 \cdot 2} = \frac{1}{2 \cdot 3} = \frac{1}{3!}$$

(Note that we used $a_2 = \frac{1}{2}a_0$ with $a_0 = 1$ to get $a_2 = \frac{1}{2}$. Furthermore, $a_1 = a_0$ with $a_0 = 1$ gives $a_1 = 1$.) In general, (verify)

$$a_{s+1} = \frac{1}{(s+1)!}$$

Hence

$$\begin{aligned} y_1 &= a_0x^0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \cdots \\ &= 1 \cdot x^0 + 1 \cdot x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \\ &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \\ &= e^x. \end{aligned}$$

Second solution. We get a second independent solution by reduction of order (Sec. 2.1). We write the ODE in standard form:

$$y'' + \frac{1-2x}{x}y' + \frac{x-1}{x}y = 0.$$

In (2), of Sec. 2.1, we then have

$$p = \frac{1-2x}{x}.$$

Hence

$$-\int p\,dx = \left[\int \frac{1}{x}dx - 2\int dx\right] = -\ln|x| + 2x = \ln\left|\frac{1}{x}\right| + 2x.$$

Applying the exponential function we get

$$\exp\left(-\int p\,dx\right) = \exp\left(\ln\left|\frac{1}{x}\right| + 2x\right) = e^{\ln|1/x|+2x} = e^{\ln|1/x|}e^{2x} = \frac{1}{x}e^{2x}.$$

Hence

$$U = \frac{1}{(e^x)^2}\left[\frac{1}{x}e^{2x}\right] = \frac{1}{e^{2x}}\left[\frac{1}{x}e^{2x}\right] = \frac{1}{x}.$$

Thus

$$u = \int U\,dx = \int \frac{1}{x}\,dx = \ln|x|.$$

From this we get

$$y_2 = uy_1 = (\ln|x|)(e^x) = e^x \ln|x|.$$

15. Hypergeometric ODE. We want to find the values of the parameters a, b, c of the given hypergeometric ODE:

$$2x(1-x)y'' + (1+6x)y' - 2y = 0.$$

Dividing the ODE by 2 gives

$$x(1-x)y'' + \left(-\tfrac{1}{2} - 3x\right)y' - y = 0.$$

The first coefficient $x(1-x)$ has the same form as in (15) on p. 186, so that you can immediately compare the coefficients of y',

$$\text{(A)} \qquad -\tfrac{1}{2} - 3x = c - (a+b+1)x$$

and the coefficients of y,

$$\text{(B)} \qquad -ab = -1.$$

From (B) we get $b = 1/a$. Substitute this into (A), obtaining from the terms in x and from the constant terms

$$a+b+1 = a + \frac{1}{a} + 1 = 3, \quad a + \frac{1}{a} = 2, \quad \text{and} \quad c = -\frac{1}{2}.$$

Hence

$$a^2 + 1 = 2a, \quad a^2 - 2a + 1 = 0, \quad (a-1)(a-1) = 0.$$

Hence $a = 1$ so that $b = \frac{1}{a} = 1$. Consequently, a first solution is

$$F(a, b, c; x) = F(1, 1, -\tfrac{1}{2}; x).$$

A second solution is given on p. 186 of the textbook by (17) and the equality that follows with

$$r_2 = 1 - c = 1 - \left(-\tfrac{1}{2}\right) = \tfrac{3}{2}.$$

that is,

$$y_2(x) = x^{1-c}F(a-c+1, b-c+1, 2-c : x)$$

Now $a - c + 1 = 1 - \left(-\frac{1}{2}\right) + 1 = \frac{5}{2}$, $b - c + 1 = 1 - \left(-\frac{1}{2}\right) + 1 = \frac{5}{2}$, and $2 - c = 2 - \left(-\frac{1}{2}\right) = \frac{5}{2}$. Thus

$$y_2(x) = x^{3/2}F\left(\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}; x\right).$$

The general solution is

$$y = AF\left(1, 1, -\tfrac{1}{2}; x\right) + Bx^{3/2}F\left(\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}; x\right), \quad A, B \text{ constants.}$$

Sec. 5.4 Bessel's Equation. Bessel Functions $J_\nu(x)$

Here is a short outline of this long section, highlighting the main points.
Bessel's equation (see p. 187)

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \qquad (1)$$

involves a given parameter ν. Hence the solutions $y(x)$ depend on ν, and one writes $y(x) = J_\nu(x)$. Integer values of ν appear often in applications, so that they deserve the special notation of $\nu = n$. We treat these first. The Frobenius method, just as the power series method, leaves a_0 arbitrary. This would make formulas and numeric values more difficult by involving an arbitrary constant. To avoid this, a_0 is assigned a definite value, that depends on n. A relatively simple series (11), p. 189 is obtained by choosing

$$a_0 = \frac{1}{2^n n!}. \qquad (9)$$

From integer n we now turn to arbitrary ν. The choice in (9) makes it necessary to generalize the factorial function $n!$ to noninteger ν. This is done on p. 190 by the gamma function, which in turn leads to a power series times a single power x^ν, as given by (20) on p. 191.

Formulas (21a), (21b), (21c), and (21d) are the backbones of formalism for Bessel functions and are important in applications as well as in theory.

On p. 193, we show that special parametric values of ν may lead to elementary functions. This is generally true for special functions; see, for instance, Team Project 14(c) in Problem Set 5.3, on p. 186.

Finally, the last topic of this long section is concerned with finding a second linearly independent solution, as shown on pp. 194–195.

Problem Set 5.4. Page 195

5. ODE reducible to Bessel's ODE (Bessel's equation). This ODE

$$x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0$$

is particularly important in applications. The second parameter λ slips into the independent variable if you set $z = \lambda x$, hence, by the chain rule $y' = dy/dx = (dy/dz)(dz/dx) = \lambda\dot{y}$, $y'' = \lambda^2\ddot{y}$, where $\dot{} \equiv d/dz$. Substitute this to get

$$\left(\frac{z^2}{\lambda^2}\right)\lambda^2\ddot{y} + \left(\frac{z}{\lambda}\right)\lambda\dot{y} + (z^2 - \nu^2)y = 0.$$

The λ cancels. A solution is $J_\nu(z) = J_\nu(\lambda x)$.

7. ODEs reducible to Bessel's ODE. For the ODE

$$x^2y'' + xy' + \tfrac{1}{4}(x^2 - 1)y = 0$$

we proceed as follows. Using the chain rule, with $z = x/2$, we get

$$y' = \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{2}.$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{2}\frac{dy}{dz}\right) = \left[\frac{d}{dz}\left(\frac{1}{2}\frac{dy}{dz}\right)\right]\frac{dz}{dx} = \left[\frac{1}{2}\frac{d^2y}{dz^2}\right]\frac{1}{2} = \frac{1}{4}\frac{d^2y}{dz^2}.$$

Caution. *Take a moment to make sure that you **completely** understand the sophisticated use of the chain rule in determining y''.*

We substitute

$$y'' = \frac{1}{4}\frac{d^2y}{dz^2} \quad \text{and} \quad y' = \frac{1}{2}\frac{dy}{dz}$$

– just obtained – into the given ODE and get

$$x^2\frac{1}{4}\frac{d^2y}{dz^2} + x\frac{1}{2}\frac{dy}{dz} + \frac{1}{4}(x^2-1)y = 0.$$

We substitute $x = 2z$:

$$4z^2\frac{1}{4}\frac{d^2y}{dz^2} + 2z\frac{1}{2}\frac{dy}{dz} + \frac{1}{4}(4z^2-1)y = 0.$$

This simplifies to

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + \left(z^2 - \frac{1}{4}\right)y = 0.$$

But this looks precisely like Bessel's equation with $z^2 - \frac{1}{4} = z^2 - \nu^2$ so that $\nu^2 = \frac{1}{4}$ and $\nu = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$. We get

$$\begin{aligned}
y &= c_1J_{1/2}(z) + c_1J_{-1/2}(z) = c_2J_{1/2}\left(\frac{x}{2}\right) + c_2J_{-1/2}\left(\frac{x}{2}\right)\\
&= c_1\sqrt{\frac{2}{\pi z}}\sin z + c_2\sqrt{\frac{2}{\pi z}}\cos z\\
&= c_1\sqrt{\frac{2}{\pi x/2}}\sin\left(\frac{x}{2}\right) + c_2\sqrt{\frac{2}{\pi x/2}}\cos\left(\frac{x}{2}\right)\\
&= x^{-1/2}\left(\widetilde{c}_1\sin\frac{1}{2}x + \widetilde{c}_2\cos\frac{1}{2}x\right),
\end{aligned}$$

where

$$\widetilde{c}_1 = \frac{2c_1}{\sqrt{\pi}} \quad \text{and} \quad \widetilde{c}_1 = \frac{2c_2}{\sqrt{\pi}}.$$

23. Integration. Proceed as suggested on p. A12. Formula (21a), with $\nu = 1$, is $(xJ_1)' = xJ_0$, integrated $xJ_1 = \int xJ_0\,dx$, and gives (with integration by parts similar to that in Example 2, on p. 192, in the text):

$$\begin{aligned}
\int x^2J_0\,dx = \int x(xJ_0)\,dx &= x(xJ_1) - \int 1\cdot xJ_1\,dx\\
\text{(C)} \qquad &= x^2J_1 - \int xJ_1\,dx.
\end{aligned}$$

Now use (21b) with $\nu = 0$, that is, $J_0' = -J_1$ to obtain on the right side of (C)

$$\begin{aligned}
&x^2J_1 - \left(x(-J_0) - \int 1\cdot(-J_0)\,dx\right)\\
&= x^2J_1 + xJ_0 - \int J_0\,dx.
\end{aligned}$$

Your CAS will perhaps give you some other functions. However, it seems essential that everything is expressed in terms of J_n with several n because Bessel functions and their integrals are usually computed recursively from J_0 and J_1; similarly for fractional values of the parameter ν.

Sec. 5.5 Bessel Functions $Y_\nu(x)$. General Solution

The Bessel functions of the second kind are introduced in order to have a basis of solutions of Bessel's equation for *all* values of ν. Recall that integer $\nu = n$ gave trouble in that J_n and J_{-n} are linearly dependent, as was shown by formula (25) on p. 194. We discuss the case $\nu = n = 0$ first and in detail on pp. 196–198. For general ν we give just the main results (on pp. 198–200) without going into all the details.

Problem Set 5.5. Page 200

7. Further ODE's reducible to Bessel's equation. We have to transform the independent variable x by setting

$$z = \frac{kx^2}{2}$$

as well as the unknown function y by setting

$$y = \sqrt{x}u.$$

Using the chain rule, we can either perform the two transformations one after another or simultaneously. We choose the latter method as follows. We determine

$$\frac{dz}{dx} = kx,$$

which we need. Differentiation with respect to x gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}x^{-1/2}u + x^{1/2}\frac{du}{dz}\frac{dz}{dx}\\ &= \frac{1}{2}x^{-1/2}u + x^{1/2}\frac{du}{dz}kx\\ &= \frac{1}{2}x^{-1/2}u + kx^{3/2}\frac{du}{dz}.\end{aligned}$$

Differentiating this again, we obtain the second derivative

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{1}{4}x^{-3/2}u + \frac{1}{2}x^{-1/2}\frac{du}{dz}kx + \frac{3}{2}kx^{1/2}\frac{du}{dz} + kx^{3/2}\frac{d^2u}{dz^2}kx\\ &= -\frac{1}{4}x^{-3/2}u + 2kx^{1/2}\frac{du}{dz} + k^2x^{5/2}\frac{d^2u}{dz^2}.\end{aligned}$$

Substituting this expression for y'' as well as y into the given equation and dividing the whole equation by $k^2x^{5/2}$ gives

$$\frac{d^2u}{dz^2} + \frac{2}{kx^2}\frac{du}{dz} + \left(1 - \frac{1}{4k^2x^4}\right)u = 0.$$

Now we recall that

$$\frac{kx^2}{2} = z \quad \text{so that } kx^2 = 2z.$$

We substitute this into the last equation and get

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{1}{16z^2}\right)u = 0.$$

This is Bessel's equation with parameter $\nu = \frac{1}{4}$. Hence a general solution of the given equation is

$$y = x^{1/2}u(z) = x^{1/2}(AJ_{1/4}(z) + BY_{1/4}(z)) = x^{1/2}\left(AJ_{1/4}\left(\frac{kx^2}{2}\right) + BY_{1/4}\left(\frac{kx^2}{2}\right)\right).$$

This corresponds to the answer on p. A13, where our constants A and B correspond to their constants c_1 and c_2. (You always can choose your constants.)

11. Hankel functions. We show linear independence, by starting from

$$c_1H_\nu^{(1)} + c_2H_\nu^{(2)} = 0.$$

We insert the definitions of the Hankel functions:

$$c_1(J_\nu + iY_\nu) + c_2(J_\nu - iY_\nu) = (c_1 + c_2)J_\nu + (ic_1 - ic_2)Y_\nu = 0.$$

Since J_ν and Y_ν are linearly independent, their coefficients must be zero, that is (divide the second coefficient by i):

$$c_1 + c_2 = 0,$$
$$c_1 - c_2 = 0.$$

Hence $c_1 = 0$ and $c_2 = 0$, which means linear independence of $H_\nu^{(1)}$ and $H_\nu^{(2)}$ on any interval on which these functions are defined.

For a second proof, as hinted on p. A13, set $H_\nu^{(1)} = kH_\nu^{(2)}$ and use (10), p. 199, to obtain a contradiction.

Chap. 6 Laplace Transforms

Laplace transforms are an essential part of the mathematical background required by engineers, mathematicians, and physicists. They have many applications in physics and engineering (electrical networks, springs, mixing problems; see Sec. 6.4). They make solving linear ODEs, IVPs (both in Sec. 6.2), systems of ODEs (Sec. 6.7), and integral equations (Sec. 6.5) easier and thus form a fitting end to Part A on ODEs. In addition, they are superior to the classical approach of Chap. 2 because they allow us to solve new problems that involve discontinuities, short impulses, or complicated periodic functions. Phenomena of "short impulses" appear in mechanical systems hit by a hammerblow, airplanes making a "hard" landing, tennis balls hit by a racket, and others (Sec. 6.4).

Two elementary examples on Laplace Transforms

Example A. Immediate Use of Table 6.1, p. 207. To provide you with an *informal introductory flavor to this chapter* and give you an *essential idea of using Laplace transforms*, take a close look at **Table 6.1** on p. 207. In the second column you see functions $f(t)$ and in the third column their Laplace transforms $\mathcal{L}(f)$, which is a convenient notational short form for $\mathcal{L}\{f(t)\}$. These Laplace transforms have been computed, are on your CAS, and become part of the background material of solving problems involving these transforms. You use the table as a "look-up" table **in both directions**, that is, in the "forward" direction starting at column 2 and ending up at column 3 via the Laplace transform $\mathcal{L}$ and in the "backward" direction from column 3 to column 2 via the inverse Laplace transform $\mathcal{L}^{-1}$. (The terms "forward" and "backward" are not standard but are only part of our informal discussion). For illustration, consider, say entry 8 in the forward direction, that is, transform

$$f(t) = \sin \omega t$$

by the **Laplace transform** $\mathcal{L}$ into

$$F(s) = \mathcal{L}(f) = \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}.$$

Using entry 8 in the backward direction, you can take the entry in column 3

$$F(s) = \frac{\omega}{s^2 + \omega^2}$$

and apply the **inverse Laplace transform** $\mathcal{L}^{-1}$ to obtain the entry in column 2:

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\} = \sin \omega t = f(t).$$

Example B. Use of Table 6.1 with Preparatory Steps. Creativity. *In many problems your creativity is required to identify which $F(s) = \mathcal{L}(f)$ in the third column provides the **best match** to the function F related to your problem.* Since the match is usually not perfect, you will learn several techniques on how to manipulate the function related to your problem. For example, if

$$F(s) = \frac{21}{s^2 + 9},$$

we see that the best match is in the third column of entry 8 and is $\omega/(s^2 + \omega^2)$.

However, the match is not perfect. Although $9 = 3^2 = \omega^2$, the numerator of the fraction is not ω but $7 \cdot \omega = 7 \cdot 3 = 21$. Therefore, we can write

$$F(s) = \frac{21}{s^2+9} = \frac{7 \cdot 3}{s^2+3^2} = 7 \cdot \frac{3}{s^2+3^2} = 7 \cdot \frac{\omega}{s^2+\omega^2}, \qquad \text{where } \omega = 3.$$

Now we have "perfected" the match and can apply the inverse Laplace transform, that is, by entry 8 backward and, in addition, linearity (as explained after the formula):

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{21}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{7 \cdot \frac{3}{s^2+3^2}\right\} = 7 \cdot \mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = 7 \cdot \sin 3t.$$

Note that since the Laplace transform is **linear** (p. 206) we were allowed to pull out the constant 7 in the third equality. Linearity is one of several techniques of Laplace transforms. *Take a moment to look over our two elementary examples that give a basic feel of the new subject.*

We introduce the techniques of Laplace transforms gradually and step by step. It will take some time and practice (with paper and pencil or typing into your computer, *without* the use of CAS, which can do most of the transforms) to get used to this new algebraic approach of Laplace transforms. Soon you will be proficient. Laplace transforms will appear again at the end of Chap. 12 on PDEs.

From calculus you may want to review **integration by parts** (used in Sec. 6.1) and, more importantly, **partial fractions** (used particularly in Secs. 6.1, 6.3, 6.5, 6.7). Partial fractions are often our last resort when more elegant ways to crack a problem escape us. Also, you may want to browse pp. A64–A66 in Sec. A3.1 on **formulas for special functions** in App. 3. All these methods from calculus are well illustrated in our carefully chosen solved problems.

Sec. 6.1 Laplace Transform. Linearity. First Shifting Theorem (s-Shifting)

This section covers several topics. It begins with the definition (1) on p. 204 of the Laplace transform, the inverse Laplace transform (1*), discusses what *linearity* means on p. 206, and derives, in Table 6.1, on p. 207, a dozen of the simplest transforms that you will need throughout this chapter and will probably have memorized after a while. Indeed, Table 6.1, p. 207, is fundamental to this chapter as was illustrated in Examples A and B in our introduction to Chap. 6.

The next important topic is about damped vibrations $e^{at}\cos\omega t$, $e^{at}\sin\omega t$ $(a < 0)$, which are obtained from $\cos\omega t$ and $\sin\omega t$ by the so-called s-shifting (Theorem 2, p. 208). *Keep linearity and s-shifting firmly in mind as they are two very important tools that already let you determine many different Laplace transforms that are variants of the ones in Table 6.1. In addition, you should know partial fractions from calculus as a third tool.*

The last part of the section on existence and uniqueness of transforms is of lesser practical interest. Nevertheless, we should recognize that, on the one hand, the Laplace transform is very general, so that even discontinuous functions have a Laplace transform; this accounts for the superiority of the method over the classical method of solving ODEs, as we will show in the next sections. On the other hand, not every function has a Laplace transform (see Theorem 3, p. 210), but this is of minor practical interest.

Problem Set 6.1. Page 210

1. Laplace transform of basic functions. Solutions by Table 6.1 and from first principles. We can find the Laplace transform of $f(t) = 3t + 12$ in two ways:

Solution Method 1. By using Table 6.1, p. 207 (which is the usual method):

$$\begin{aligned}\mathcal{L}(3t+12) &= 3\mathcal{L}(t) + 12\mathcal{L}(1)\\ &= 3\cdot\frac{1}{s^2} + 12\cdot\frac{1}{s}\\ &= \frac{3}{s^2} + \frac{12}{s} \qquad \text{(Table 6.1, p. 207; entries 1 and 2).}\end{aligned}$$

Solution Method 2. From the definition of the Laplace transform (i.e., from first principles). The purpose of solving the problem from scratch ("first principles") is to give a better understanding of the definition of the Laplace transform, as well as to illustrate how one would derive the entries of Table 6.1—or even as more complicated transforms. Example 4 on pp. 206–207 of the textbook shows our approach for cosine and sine. (The philosophy of our approach is similar to the approach used at the beginning of calculus, when the derivatives of elementary functions were determined directly from the definition of a derivative of a function at a point.)

From the definition (1) of the Laplace transform and by calculus we have

$$\mathcal{L}(3t+12) = \int_0^\infty te^{-st}(3t+12)\,dt = 3\int_0^\infty te^{-st}t\,dt + 12\int_0^\infty e^{-st}\,dt.$$

We solve each of the two integrals just obtained separately. The second integral of the last equation is easier to solve and so we do it first:

$$\begin{aligned}\int_0^\infty e^{-st}\,dt &= \lim_{T\to\infty}\int_0^T e^{-st}\,dt\\ &= \lim_{T\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^T\\ &= \lim_{T\to\infty}\left[-\frac{1}{s}e^{-sT}\right] - \left(-\frac{1}{s}e^{-s\cdot 0}\right)\\ &= 0 + \frac{1}{s} \qquad (\text{since } e^{-st}\to 0 \text{ as } T\to\infty \text{ for } s>0).\end{aligned}$$

Thus we obtain

$$12\int_0^\infty e^{-st}\,dt = 12\cdot\frac{1}{s}.$$

The first integral is

$$\int_0^\infty te^{-st}t\,dt = \lim_{T\to\infty}\int_0^T te^{-st}\,dt.$$

Brief review of the method of ***integration by parts*** *from calculus*. Recall that **integration by parts** is

$$\int uv'dx = uv - \int u'v\,dx \qquad \text{(see inside covers of textbook).}$$

We apply the method to the indefinite integral

$$\int te^{-st}\,dt.$$

In the method we have to decide to what we set u and v' equal. The goal is to make the second integral involving $u'v$ simpler than the original integral. (There are only two choices and if we make the wrong choice, thereby making the integral more difficult, then we pick the second choice.) If the integral involves a polynomial in t, setting u to be equal to the polynomial is a good choice (differentiation will reduce the degree of the polynomial). We choose

$$u = t, \qquad \text{then} \qquad u' = 1;$$

and

$$v' = e^{-st}, \qquad \text{then} \qquad v = \int e^{-st}\,dt = \frac{e^{-st}}{-s}.$$

Then

$$\begin{aligned}\int te^{-st}\,dt &= t\cdot\frac{e^{-st}}{-s} - \int 1\cdot\frac{e^{-st}}{-s}\,dt\\ &= -\frac{te^{-st}}{s} + \frac{1}{s}\int e^{-st}\,dt\\ &= -\frac{te^{-st}}{s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right) = -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \qquad \text{(with constant } C \text{ of integration set to 0).}\end{aligned}$$

Going back to our original problem (the first integral) and using our result just obtained

$$\begin{aligned}\lim_{T\to\infty}\int_0^T te^{-st}\,dt &= \lim_{T\to\infty}\left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^T\\ \text{(A)} \qquad &= \lim_{T\to\infty}\left[-\frac{Te^{-sT}}{s} - \frac{e^{-sT}}{s^2}\right] - \left[-\frac{0e^{-s\cdot 0}}{s} - \frac{e^{-s\cdot 0}}{s^2}\right].\end{aligned}$$

Now since

$$e^{sT} \gg T \qquad \text{for } s > 0,$$

it follows that

$$\lim_{T\to\infty}\left[-\frac{Te^{-sT}}{s}\right] = 0. \qquad \text{Also} \qquad \lim_{T\to\infty}\left[-\frac{e^{-sT}}{s^2}\right] = 0.$$

So, when we simplify all our calculations, recalling that $e^0 = 1$, we get that (A) equals $1/s^2$. Hence

$$3\int_0^\infty te^{-st}t\,dt = 3\frac{1}{s^2}.$$

Thus, having solved the two integrals, we obtain (as before)

$$\mathcal{L}(3t+12) = 3\int_0^\infty te^{-st}t\,dt + 12\int_0^\infty e^{-st}\,dt = 3\frac{1}{s^2} + 12\cdot\frac{1}{s}, \qquad s > 0.$$

7. Laplace transform. Linearity. ***Hint.*** *Use the addition formula for the sine* (see (6), p. A64). This gives us

$$\sin(\omega t + \theta) = \sin \omega t \cos \theta + \cos \omega t \sin \theta.$$

Next we apply the Laplace transform to each of the two terms on the right, that is,

$$\begin{aligned}
\mathcal{L}(\sin \omega t \cos \theta) &= \int_0^\infty e^{-st} \sin \omega t \cos \theta \, dt \\
&= \cos \theta \int_0^\infty e^{-st} \sin \omega t \, dt && \text{(Theorem 1 on p. 206 for “pulling apart”)} \\
&= \cos \theta \frac{\omega}{s^2 + \omega^2} && \text{(Table 6.1 on p. 207, entry 8).} \\
\mathcal{L}(\cos \omega t \cos \theta) &= \sin \theta \frac{s}{s^2 + \omega^2} && \text{(Theorem 1 and Table 6.1, entry 7).}
\end{aligned}$$

Together, using Theorem 1 (for “putting together”) we obtain

$$\mathcal{L}(\sin(\omega t + \theta)) = \mathcal{L}(\sin \omega t \cos \theta) + \mathcal{L}(\cos \omega t \cos \theta) = \frac{\omega \cos \theta + \sin \theta}{s^2 + \omega^2}.$$

11. Use of the integral that defines the Laplace transform. The function, shown graphically, consists of a line segment going from $(0, 0)$ to (b, b) and then drops to 0 and stays 0. Thus,

$$f(t) = \begin{cases} t & \text{if } 0 \le t \le b \\ 0 & \text{if } t > b. \end{cases}$$

The latter part, where f is always 0, does not contribute to the Laplace transform. Hence we consider only

$$\begin{aligned}
\int_0^b te^{-st} \, dt &= \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^b && \text{(by Prob. 1, Sec. 6.1, second solution from before)} \\
&= -\frac{be^{-sb}}{s} - \frac{e^{-sb}}{s^2} + \frac{1}{s^2} = \frac{1 - e^{-bs}}{s^2} - \frac{be^{-bs}}{s}.
\end{aligned}$$

Remark about solving **Probs. 9–16**. At first you have to express algebraically the function that is depicted, then you use the integral that defines the Laplace transform.

21. Nonexistence of the Laplace transform. For instance, e^{t^2} has no Laplace transform because the integrand of the defining integral is $e^{t^2} e^{-st} = e^{t^2 - st}$ and $t^2 - st > 0$ for $t > s$, and the integral from 0 to ∞ of an exponential function with a positive exponent does not exist, that is, it is infinite.

25. Inverse Laplace transform. First we note that $3.24 = (1.8)^2$. Using Table 6.1 on p. 207 backwards, that is, “which $\mathcal{L}(f)$ corresponds to f,” we get

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{0.2s + 1.8}{s^2 + 3.24}\right) &= \mathcal{L}^{-1}\left(0.2 \frac{s}{s^2 + (1.8)^2} + \frac{1.8}{s^2 + (1.8)^2}\right) \\
&= 0.2\mathcal{L}^{-1}\left(\frac{s}{s^2 + (1.8)^2}\right) + \mathcal{L}^{-1}\left(\frac{1.8}{s^2 + (1.8)^2}\right) = 0.2 \cos 1.8t + \sin 1.8t.
\end{aligned}$$

In **Probs. 29–40**, use Table 6.1 (p. 207) and, in some problems, also use reduction by partial fractions. When using Table 6.1 and looking at the $\mathcal{L}(f)$ column, also think about linearity, that is, from **Prob. 24** for the inverse Laplace transform,

$$\mathcal{L}^{-1}(\mathcal{L}(af) + \mathcal{L}(bg)) = a\mathcal{L}^{-1}(\mathcal{L}(f)) + b\mathcal{L}^{-1}(\mathcal{L}(f)) = af + bg.$$

Furthermore, you may want to look again at Examples A and B at the opening of Chap. 6 of this Study Guide.

29. Inverse transform. We look at Table 6.1, p. 207, and find, under the $\mathcal{L}(f)$ column, the term that matches most closely to what we are given in the problem. Entry 4 seems to fit best, that is:

$$\frac{n!}{s^{n+1}}.$$

We are given

$$\frac{12}{s^4} - \frac{228}{s^6}.$$

Consider

$$\frac{12}{s^4}. \qquad \text{For } n = 3 \qquad \frac{n!}{s^{n+1}} = \frac{3!}{s^{3+1}} = \frac{3 \cdot 2 \cdot 1}{s^{3+1}} = \frac{6}{s^{3+1}}.$$

Thus

$$\frac{12}{s^4} = 2 \cdot \frac{3!}{s^{3+1}}.$$

Consider

$$\frac{228}{s^6}. \qquad \text{For } n = 5 \qquad \frac{n!}{s^{n+1}} = \frac{5!}{s^{5+1}} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{s^{5+1}} = \frac{120}{s^{5+1}}.$$

Now

$$228 = 2 \cdot 2 \cdot 3 \cdot 19,$$
$$5! = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5.$$

Thus, the greatest common divisor (gcd) of 228 and 5! is

$$\gcd(228, 5!) = 2 \cdot 2 \cdot 3.$$

Hence,

$$228 = (2 \cdot 2 \cdot 2 \cdot 3 \cdot 5) \cdot 19 \cdot \frac{1}{2 \cdot 5} = \frac{19}{10} \cdot 5!$$

so that

$$\frac{228}{s^6} = \frac{19}{10} \cdot \frac{5!}{s^{5+1}}.$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{12}{s^4} - \frac{228}{s^6}\right) = 2\mathcal{L}^{-1}\left(\frac{3!}{s^{3+1}}\right) - \frac{19}{10}\mathcal{L}^{-1}\left(\frac{5!}{s^{5+1}}\right) = 2t^3 - \frac{19}{10}t^5,$$

which corresponds to the answer on p. A13.

39. First shifting theorem. The first shifting theorem on p. 208 states that, under suitable conditions (for details see Theorem 2, p. 208),

$$\text{if} \qquad \mathcal{L}\{f(t)\} = F(s), \qquad \text{then} \qquad \mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We want to find

$$\mathcal{L}^{-1}\left(\frac{21}{(s+\sqrt{2})^4}\right).$$

From entry 4 of Table 6.1, p. 207, we know that

$$\mathcal{L}^{-1}\left(\frac{3!}{s^{3+1}}\right) = t^3.$$

Hence, by Theorem 2,

$$\mathcal{L}^{-1}\left(\frac{21}{(s+\sqrt{2})^4}\right) = t^3 e^{-\sqrt{2}t} \cdot \frac{21}{3!} = \frac{7}{2} \cdot t^3 e^{-\sqrt{2}t}.$$

Sec. 6.2 Transforms of Derivatives and Integrals. ODEs

The main purpose of the Laplace transform is to solve differential equations, mainly ODEs. The heart of Laplacian theory is that we first transform the given ODE (or an initial value problem) into an equation (*subsidiary equation*) that involves the **Laplace transform.** Then we solve the subsidiary equation by algebra. Finally, we use the **inverse Laplace transform** to transform the solution of the subsidiary equation back into the solution of the given ODE (or initial value problem). Figure 116, p. 215, shows the steps of the transform method. In brief, we go from t-space to s-space by $\mathcal{L}$ and go back to t-space by $\mathcal{L}^{-1}$. Essential to this method is Theorem 1, p. 211, on the transforms of derivatives, because it allows one to solve ODEs of first order [by (1)] and second order [by (2)]. Of lesser immediate importance is Theorem 3, p. 213, on the transforms of integrals.

Note that such transform methods appear throughout mathematics. They are important *because they allow us to transform (convert) a* ***hard or even impossible*** *problem* from the "original space" *to* ***another (easier)*** *problem* in "another space," which we can solve, and then transform the solution from that "other space" back to the "original space." Perhaps, the simplest example of such an approach is logarithms (see beginning of Sec. 6.2 on p. 211).

Problem Set 6.2. Page 216

5. Initial value problem. *First, by the old method of Sec. 2.2, pp. 53–61.* The problem $y'' - \frac{1}{4}y = 0$, $y(0) = 12$, $y'(0) = 0$ can readily be solved by the method in Sec. 2.2. A general solution is

$$y = c_1 e^{t/2} + c_2 e^{-t/2}, \qquad \text{and} \qquad y(0) = c_1 + c_2 = 12.$$

We need the derivative

$$y' = \tfrac{1}{2}(c_1 e^{t/2} - c_2 e^{-t/2}), \qquad \text{and} \qquad y'(0) = \tfrac{1}{2}(c_1 - c_2) = 0.$$

From the second equation for c_1 and c_2, we have $c_1 = c_2$, and then $c_1 = c_2 = 6$ from the first of them. This gives the solution

$$y = 6(e^{t/2} + e^{-t/2}) = 12 \cosh \tfrac{1}{2}t \qquad \text{(see p. A13).}$$

Second, by the new method of Sec. 6.2. The point of this problem is to show how initial value problems can be handled by the transform method directly, that is, these problems do not require solving the homogeneous ODE first.

We need $y(0) = 4$, $y'(0) = 0$ and obtain from (2) the subsidiary equation

$$\mathcal{L}\left(y'' - \tfrac{1}{4}y\right) = \mathcal{L}(y'') - \tfrac{1}{4}\mathcal{L}(y) = s^2\mathcal{L}(y) - 4s - \tfrac{1}{4}\mathcal{L}(y) = 0.$$

Collecting the $\mathcal{L}(y)$-terms on the left, we have

$$\left(s^2 - \tfrac{1}{4}\right)\mathcal{L}(y) = 12s.$$

Thus, we obtain the solution of the subsidiary equation

$$Y = \mathcal{L}(y) = \frac{12s}{s^2 - \frac{1}{4}}, \qquad \text{so that} \qquad y = \mathcal{L}^{-1}(Y) = 12 \cosh \frac{1}{2}t.$$

13. Shifted data. Shifted data simply means that, if your initial values are given at a t_0 (which is different from 0), you have to set $t = \tilde{t} + t_0$, so that $t = t_0$ corresponds to $\tilde{t} = 0$ and you can apply (1) and (2) to the "shifted problem."

The problem $y' - 6y = 0$, $y(-1) = 4$ has the solution $y = 4\,e^{6(t+1)}$, as can be seen almost by inspection. To obtain this systematically by the Laplace transform, proceed as on p. 216. Set

$$t = \tilde{t} + t_0 = \tilde{t} - 1,$$

so that $\tilde{t} = t + 1$. We now have the shifted problem:

$$\tilde{y}' - 6\tilde{y} = 0, \qquad \tilde{y}(0) = 4.$$

Writing $\widetilde{Y} = \mathcal{L}(\tilde{y})$, we obtain the subsidiary equation for the shifted problem:

$$\mathcal{L}(\tilde{y}' - 6\tilde{y}) = \mathcal{L}(\tilde{y}') - 6\mathcal{L}(\tilde{y}) = s\widetilde{Y} - 4 - 6\widetilde{Y} = 0.$$

Hence

$$(s-6)\widetilde{Y} = 4, \qquad \widetilde{Y} = \frac{4}{s-6}, \qquad \tilde{y}(\tilde{t}) = 4e^{6\tilde{t}}, \qquad y(t) = 4e^{6(t+1)}.$$

17. Obtaining transforms by differentiation (Theorem 1). Differentiation is primarily for solving ODEs, but it can also be used for deriving transforms. We will succeed in the case of

$$f(t) = te^{-at}. \qquad \text{We have} \qquad f(0) = 0.$$

Then, by two differentiations, we obtain

$$f'(t) = e^{-at} + te^{-at}(-a) = e^{-at} - ate^{-at}, \qquad f'(0) = 1$$
$$\begin{aligned} f''(t) &= -ae^{-at} - (ae^{-at} + ate^{-at}(-a)) \\ &= -ae^{-at} - ae^{-at} + a^2te^{-at} = -2ae^{-at} + a^2te^{-at}. \end{aligned}$$

Taking the transform on both sides of the last equation and using linearity, we obtain

$$\begin{aligned} \mathcal{L}(f'') &= -2a\mathcal{L}(e^{-at}) + a^2\mathcal{L}(f) \\ &= -2a\frac{1}{s+a} + a^2\mathcal{L}(f). \end{aligned}$$

From (2) of Theorem 1 on p. 211 we know that

$$\begin{aligned} \mathcal{L}(f'') &= s^2\mathcal{L}(f) - sf(0) - f'(0) \\ &= s^2\mathcal{L}(f) - s \cdot 0 - 1. \end{aligned}$$

Since the left-hand sides of the two last equations are equal, their right-hand sides are equal, that is,

$$\begin{aligned} s^2\mathcal{L}(f) - 1 &= -2a\frac{1}{s+a} + a^2\mathcal{L}(f), \\ s^2\mathcal{L}(f) - a^2\mathcal{L}(f) &= -2a\frac{1}{s+a} + 1. \end{aligned}$$

However,

$$-2a\frac{1}{s+a} + 1 = \frac{-2a + (s+a)}{s+a} = \frac{s-a}{s+a},$$
$$\mathcal{L}(f)(s^2 - a^2) = \frac{s-a}{s+a}.$$

From this we obtain,

$$\begin{aligned} \mathcal{L}(f) &= \frac{s-a}{s+a} \cdot \frac{1}{s^2 - a^2} \\ &= \frac{s-a}{s+a} \cdot \frac{1}{(s+a)(s-a)} = \frac{1}{(s+a)^2}. \end{aligned}$$

23. Application of Theorem 3. We have from Table 6.1 in Sec. 6.1

$$\mathcal{L}^{-1}\left(\frac{1}{s + \frac{1}{4}}\right) = e^{-t/4}.$$

By linearity

$$\mathcal{L}^{-1}\left(\frac{3}{s^2 + \frac{s}{4}}\right) = 3\mathcal{L}^{-1}\left(\frac{1}{s^2 + \frac{s}{4}}\right).$$

Now

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{s^2+\frac{s}{4}}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s\left(s+\frac{1}{4}\right)}\right) \\
&= \mathcal{L}^{-1}\left(\frac{1}{s}\frac{1}{s+\frac{1}{4}}\right) \\
&= \int_0^t e^{-\tau/4}\,d\tau \qquad \left[\begin{array}{l}\text{by Theorem 3, p. 213, second integral formula,} \\ \text{with } F(s) = \dfrac{1}{s+\frac{1}{4}}\end{array}\right] \\
&= -4e^{-t/4}\Big|_0^t = 4 - 4\,e^{-t/4}.
\end{aligned}
$$

Hence

$$\mathcal{L}^{-1}\left(\frac{3}{s^2+\frac{s}{4}}\right) = 3\cdot(4-4e^{-t/4}) = 12 - 12e^{-t/4}.$$

Sec. 6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

The great advantage of the Laplace transformation method becomes fully apparent in this section, where we encounter a powerful auxiliary function, the **unit step function** or **Heaviside** function $u(t-a)$. It is defined by (1), p. 217. Take a good look at **Figs. 118**, **119**, and **120** to understand the "turning on/off" and "shifting" effects of that function when applied to other functions. It is made for engineering applications where we encounter periodic phenomena.

An overview of this section is as follows. **Example 1**, p. 220, shows how to represent a piecewise given function in terms of unit step functions, which is simple, and how to obtain its transform by the **second shifting theorem** (Theorem 1 on p. 219), which needs patience (see also below for more details on Example 1).

Example 2, p. 221, shows how to obtain inverse transforms; here the exponential functions indicate that we will obtain a piecewise given function in terms of unit step functions (see Fig. 123 on p. 221 of the book).

Example 3, pp. 221–222, shows the solution of a first-order ODE, the model of an *RC*-circuit with a "piecewise" electromotive force.

Example 4, pp. 222–223, shows the same for an *RLC*-circuit (see Sec. 2.9), whose model, when differentiated, is a second-order ODE.

More details on Example 1, p. 220. Application of Second Shifting Theorem (Theorem 1). We consider $f(t)$ term by term and think over what must be done. Nothing needs to be done for the first term of f, which is 2. The next part, $\frac{1}{2}t^2$, has two contributions, one involving $u(t-1)$ and the other involving $u(t-\pi/2)$. For the first contribution, we write (for direct applicability of Theorem 1)

$$
\begin{aligned}
\tfrac{1}{2}t^2\,u(t-1) &= \tfrac{1}{2}\left[(t-1)^2 + 2t - 1\right]u(t-1) \\
&= \tfrac{1}{2}\left[(t-1)^2 + 2(t-1) + 1\right]u(t-1).
\end{aligned}
$$

For the second contribution we write

$$
\begin{aligned}
-\frac{1}{2}t^2\,u\left(t-\frac{\pi}{2}\right) &= -\frac{1}{2}\left[\left(t-\frac{\pi}{2}\right)^2 + \pi t - \frac{1}{4}\pi^2\right]u\left(t-\frac{\pi}{2}\right) \\
&= -\frac{1}{2}\left[\left(t-\frac{\pi}{2}\right)^2 + \pi\left(t-\frac{\pi}{2}\right) + \frac{1}{4}\pi^2\right]u\left(t-\frac{\pi}{2}\right).
\end{aligned}
$$

Finally, for the last portion [line 3 of $f(t)$, $\cos\, t$], we write

$$\begin{aligned}(\cos\, t)\, u\Big(t-\frac{\pi}{2}\Big) &= \cos\Big(t-\frac{\pi}{2}+\frac{\pi}{2}\Big)\, u\Big(t-\frac{\pi}{2}\Big)\\ &= \Big[\cos\Big(t-\frac{\pi}{2}\Big)\cos\frac{\pi}{2} - \sin\Big(t-\frac{\pi}{2}\Big)\sin\frac{\pi}{2}\Big]\, u\Big(t-\frac{\pi}{2}\Big)\\ &= \Big[0-\sin\Big(t-\frac{\pi}{2}\Big)\Big]\, u\Big(t-\frac{\pi}{2}\Big).\end{aligned}$$

Now all the terms are in a suitable form to which we apply Theorem 1 directly. This yields the result shown on p. 220.

Problem Set 6.3. Page 223

5. Second shifting theorem. For applying the theorem, we write

$$e^t\Big[1-u\Big(t-\frac{\pi}{2}\Big)\Big] = e^t - \exp\Big[\frac{\pi}{2}+\Big(t-\frac{\pi}{2}\Big)\Big]\, u\Big(t-\frac{\pi}{2}\Big) = e^t - e^{\pi/2}e^{t-\pi/2}u\Big(t-\frac{\pi}{2}\Big).$$

We apply the second shifting theorem (p. 219) to obtain the transform

$$\frac{1}{s-1} - \frac{e^{\pi/2}e^{-(\pi/2)s}}{s-1} = \frac{1}{s-1}\Big[1-\exp\Big(\frac{\pi}{2}-\frac{\pi}{2}s\Big)\Big].$$

13. Inverse transform. We have

$$\frac{6}{s^2+9} = 2\cdot\frac{3}{s^2+3^2}.$$

Hence, by Table 6.1, p. 207, we have that the inverse transform of $6/(s^2+9)$ is $2\sin 3t$. Also

$$\frac{-6e^{-\pi s}}{s^2+9} = -2\cdot\frac{3e^{-\pi s}}{s^2+3^2}.$$

Hence, by the shifting theorem, p. 219, we have that

$$\frac{3e^{-\pi s}}{s^2+3^2} \qquad \text{has the inverse} \quad \sin 3(t-\pi)\, u(t-\pi).$$

Since

$$\sin 3(t-\pi) = -\sin 3t \qquad \text{(periodicity)}$$

we see that

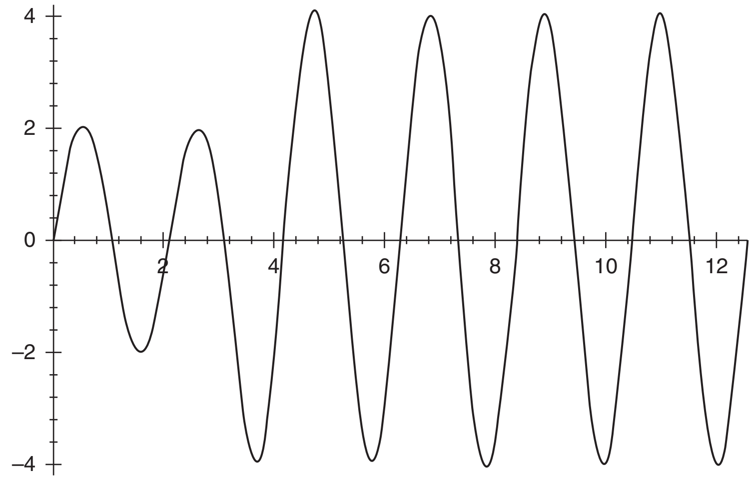
$$\sin 3(t-\pi)\, u(t-\pi) = -\sin 3t\, u(t-\pi).$$

Putting it all together, we have

$$\begin{aligned} f(t) &= 2\sin 3t - [-\sin 3t\, u(t-\pi)] = 2\sin 3t + 2\sin 3t\, u(t-\pi)\\ &= 2[1+u(t-\pi)]\sin 3t\end{aligned}$$

as given on p. A14. This means that

$$f(t) = \begin{cases} 2\sin 3t & \text{if } 0<t<\pi\\ 4\sin 3t & \text{if } t>\pi.\end{cases}$$



Sec. 6.3 Prob. 13. Inverse transform

19. Initial value problem. Solution by partial fractions. We solve this problem by partial fractions. We can always use partial fractions when we cannot think of a simpler solution method. The subsidiary equation of $y'' = 6y' + 8y = e^{-3t} - e^{-5t}$, $y(0) = 0$, $y'(0) = 0$ is

$$(s^2Y - y(0)s - y'(0)) + 6(sY - y(0)) + 8Y = \frac{1}{s-(-3)} - \frac{1}{s-(-5)}.$$

Hence

$$(s^2 + 6s + 8)Y = \frac{1}{s+3} - \frac{1}{s+5}.$$

Now $s^2 + 6s + 8 = (s+2)(s+4)$ so that

$$(s+2)(s+4)Y = \frac{1}{s+3} - \frac{1}{s+5}.$$

Solving for Y and simplifying gives

$$\begin{aligned} Y &= \frac{1}{(s+2)(s+4)(s+3)} - \frac{1}{(s+2)(s+4)(s+5)} \\ &= \frac{(s+5)-(s+3)}{(s+2)(s+3)(s+4)(s+5)} \\ &= 2 \cdot \frac{1}{(s+2)(s+3)(s+4)(s+5)}. \end{aligned}$$

We use partial fractions to express Y in a form more suitable for applying the inverse Laplace transform. We set up

$$\frac{1}{(s+2)(s+3)(s+4)(s+5)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4} + \frac{D}{s+5}.$$

To determine A, we multiply both sides of the equation by $s+2$ and obtain

$$\frac{1}{(s+3)(s+4)(s+5)} = A + \frac{B(s+2)}{s+3} + \frac{C(s+2)}{s+4} + \frac{D(s+2)}{s+5}.$$

Next we substitute $s=-2$ and get

$$\begin{aligned}\frac{1}{(s+3)(s+4)(s+5)} &= \frac{1}{(-2+1)(-2+4)(-2+3)}\\ &= A+\frac{B\cdot(-2+2)}{-2+3}+\frac{C\cdot(-2+2)}{-2+4}+\frac{D\cdot(-2+2)}{-2+5}.\end{aligned}$$

This simplifies to

$$\frac{1}{(1)(2)(3)} = A+0+0+0,$$

from which we immediately determine that

$$\boxed{A=\tfrac{1}{6}}.$$

Similarly, multiplying by $s+3$ and then substituting $s=-3$ gives the value for B:

$$\frac{1}{(s+2)(s+4)(s+5)} = \frac{A(s+3)}{s+2}+B+\frac{C(s+3)}{s+4}+\frac{D(s+3)}{s+5}, \qquad \frac{1}{(-1)(1)(2)} = \boxed{-\tfrac{1}{2}=B}.$$

For C we have

$$\frac{1}{(s+2)(s+3)(s+5)} = \frac{A(s+4)}{s+2}+\frac{B(s+4)}{s+3}+C+\frac{D(s+4)}{s+5}, \qquad \frac{1}{(-2)(-1)(1)} = \boxed{\tfrac{1}{2}=C}.$$

Finally, for D we get

$$\frac{1}{(s+2)(s+3)(s+4)} = \frac{A(s+5)}{s+2}+\frac{B(s+5)}{s+3}+\frac{C(s+5)}{s+4}+D, \qquad \frac{1}{(-3)(-2)(-1)} = \boxed{-\tfrac{1}{6}=D}.$$

Thus

$$\begin{aligned}\frac{1}{(s+2)(s+3)(s+4)(s+5)} &= \frac{A}{s+2}+\frac{B}{s+3}+\frac{C}{s+4}+\frac{D}{s+5}\\ &= \frac{\frac{1}{6}}{s+2}+\frac{-\frac{1}{2}}{s+3}+\frac{\frac{1}{2}}{s+4}+\frac{-\frac{1}{6}}{s+5}.\end{aligned}$$

This means that Y can be expressed in the following form:

$$Y = 2\cdot\left[\frac{\frac{1}{6}}{s+2}+\frac{-\frac{1}{2}}{s+3}+\frac{\frac{1}{2}}{s+4}+\frac{-\frac{1}{6}}{s+5}\right] = \frac{\frac{1}{3}}{s+2}+\frac{-1}{s+3}+\frac{1}{s+4}+\frac{-\frac{1}{3}}{s+5}.$$

Using linearity and Table 6.1, Sec. 6.1, we get

$$\begin{aligned}y = \mathcal{L}^{-1}(Y) &= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)-\mathcal{L}^{-1}\left(\frac{1}{s+3}\right)+\mathcal{L}^{-1}\left(\frac{1}{s+4}\right)-\frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)\\ &= \frac{1}{3}e^{-2t}-e^{-3t}+e^{-4t}-\frac{1}{3}e^{-5t}.\end{aligned}$$

This corresponds to the answer given on p. A14, because we can write

$$\tfrac{1}{3}e^{-2t} - e^{-3t} + e^{-4t} - \tfrac{1}{3}e^{-5t} = \tfrac{1}{3}e^{-5t}(3e^{t} - 3e^{2t} + e^{3t} - 1) = \tfrac{1}{3}(e^{t} - 1)^3 e^{-5t}.$$

21. Initial value problem. Use of unit step function. The problem consists of the ODE

$$y'' + 9y = \begin{cases} 8\sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } t > \pi \end{cases}$$

and the initial conditions $y(0) = 0$, $y'(0) = 4$. It has the subsidiary equation

$$\begin{aligned} s^2Y - 0 \cdot s - 4 + 9Y &= 8\mathcal{L}[\sin t - u(t-\pi)\sin t] \\ &= 8\mathcal{L}[\sin t + u(t-\pi)\sin(t-\pi)] \\ &= 8(1 + e^{-\pi s})\frac{1}{s^2+1}, \end{aligned}$$

simplified

$$(s^2 + 9)Y = 8(1 + e^{-\pi s})\frac{1}{s^2+1} + 4.$$

The solution of this subsidiary equation is

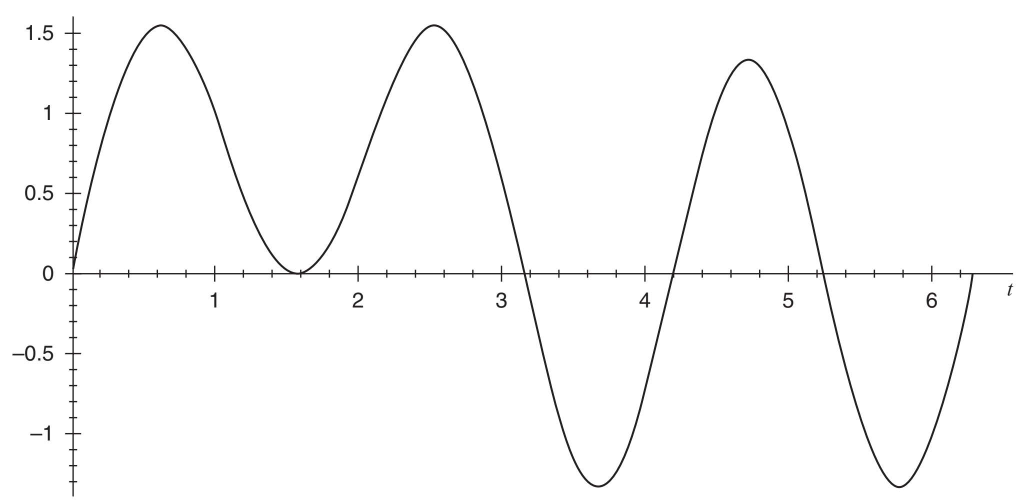
$$Y = \frac{8(1 + e^{-\pi s})}{(s^2+9)(s^2+1)} + \frac{4}{s^2+9}.$$

Apply partial fraction reduction

$$\frac{8}{(s^2+9)(s^2+1)} = \frac{1}{s^2+1} - \frac{1}{s^2+9}.$$

Since the inverse transform of $4/(s^2 + 9)$ is $\frac{4}{3}\sin 3t$, we obtain

$$y = \mathcal{L}^{-1}(Y) = \sin t - \tfrac{1}{3}\sin 3t + \left[\sin(t-\pi) - \tfrac{1}{3}\sin(3(t-\pi))\right]u(t-\pi) + \tfrac{4}{3}\sin 3t.$$



Sec. 6.3 Prob. 21. Solution of the initial value problem

Hence, if $0 < t < \pi$, then

$$y(t) = \sin t + \sin 3t$$

and if $t > \pi$, then

$$y(t) = \tfrac{4}{3}\sin 3t.$$

39. ***RLC*-circuit.** The model is, as explained in Sec. 2.9, pp. 93–97, and in Example 4, pp. 222–223, where i is the **current** measured in amperes (A).

$$i' + 2i + 2\int_0^t i(\tau)\,d\tau = 1000(1 - u(t-2)).$$

The factor 1000 comes in because v is 1 kV = 1000 V. If you solve the problem in terms of kilovolts, then you don't get that factor! Obtain its subsidiary equation, noting that the initial current is zero,

$$sI + 2I + \frac{2I}{s} = 1000 \cdot \frac{1 - e^{-2s}}{s}.$$

Multiply it by s to get

$$(s^2 + 2s + 2)I = 1000(1 - e^{-2s}).$$

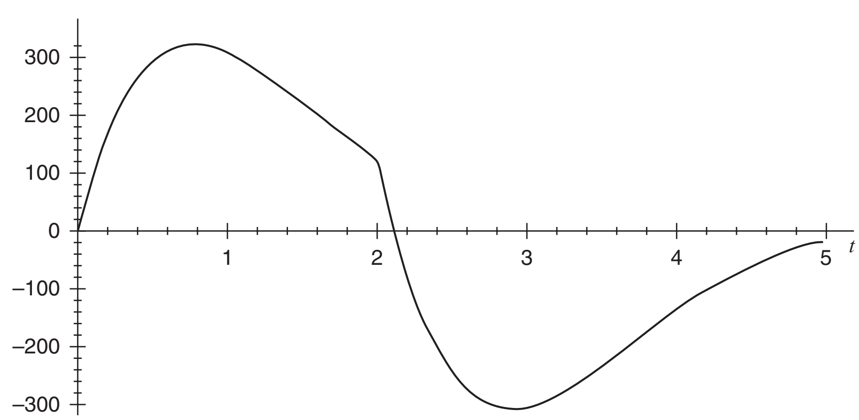
Solve it for I:

$$I = \frac{1000(1 - e^{-2s})}{(s+1)^2 + 1} = \frac{1000(1 - e^{-2s})}{s^2 + 2s + 2}.$$

Complete the solution by using partial fractions from calculus, as shown in detail in the textbook in Example 4, pp. 222–223. Having done so, obtain its inverse (the solution $i(t)$ of the problem):

$$i = \mathcal{L}^{-1}(I) = 1000 \cdot e^{-t}\sin t - 1000 \cdot u(t-2) \cdot e^{-(t-2)}\sin(t-2), \quad \text{measured in amperes (A).}$$

This equals $1000e^{-t}\sin t$ if $0 < t < 2$ and $1000e^{-t}\sin t - e^{-(t-2)}\sin(t-2)$ if $t > 2$. See the accompanying figure. Note the discontinuity of i' at $t = 2$ corresponding to the discontinuity of the electromotive force on the right side of the ODE (rather: of the integro-differential equation).



Sec. 6.3 Prob. 39. Current $i(t)$ in amperes (A)

Sec. 6.4 Short Impulses. Dirac's Delta Function. Partial Fractions.

The next function designed for engineering applications is the important **Dirac delta function** $\delta(t-a)$ defined by (3) on p. 226. Together with Heaviside's step function (Sec. 6.3), they provide powerful tools for applications in mechanics, electricity, and other areas. They allow us to solve problems that could not be solved with the methods of Chaps. 1–5. Note that we have used partial fractions earlier in this chapter and will continue to do so in this section when solving problems of forced vibrations.

Problem Set 6.4. Page 230

3. Vibrations. This initial value problem

$$y'' + y = \delta(t-\pi), \qquad y(0) = 8, \qquad y'(0) = 0$$

models an undamped motion that starts with initial displacement 8 and initial velocity 0 and receives a hammerblow at a later instant (at $t = \pi$). Obtain the subsidiary equation

$$s^2Y - 8s + 4Y = e^{-\pi s}, \qquad \text{thus} \qquad (s^2+4)Y = e^{-\pi s} + 8s.$$

Solve it:

$$Y = \frac{8s}{s^2+2^2} + \frac{e^{-\pi s}}{s^2+2^2}.$$

Obtain the inverse, giving the motion of the system, the displacement, as a function of time t,

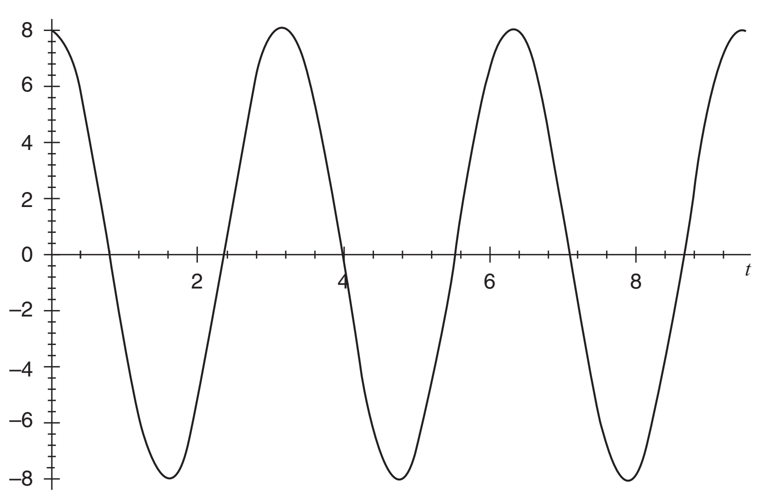
$$y = \mathcal{L}^{-1}(Y) = 8\mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right) + \mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+2^2}\right).$$

Now

$$\mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+2^2}\right) = \sin(2(t-\pi)) \cdot \frac{1}{2}u(t-\pi).$$

From the periodicity of sin we know that

$$\sin(2(t-\pi)) = \sin(2t-2\pi) = \sin 2t.$$



Sec. 6.4 Prob. 3. Vibrations

Hence the final answer is

$$y = 8\ \cos 2t + (\sin 2t)\tfrac{1}{2}u(t-\pi),$$

given on p. A14 and shown in the accompanying figure, where the effect of $\sin 2t$ (beginning at $t = \pi$) is hardly visible.

5. Initial value problem. This is an undamped forced motion with two impulses (at $t = \pi$ and 2π) as the driving force:

$$y'' + y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 1.$$

By Theorem 1, Sec. 6.2, and (5), Sec. 6.4, we have

$$(s^2 Y - s \cdot 0 - 1) + Y = e^{-\pi s} - e^{-2\pi s}$$

so that

$$(s^2 + 1)Y = e^{-\pi s} - e^{-2\pi s} + 1.$$

Hence,

$$Y = \frac{1}{s^2+1}(e^{-\pi s} - e^{-2\pi s} + 1).$$

Using linearity and applying the inverse Laplace transform to each term we get

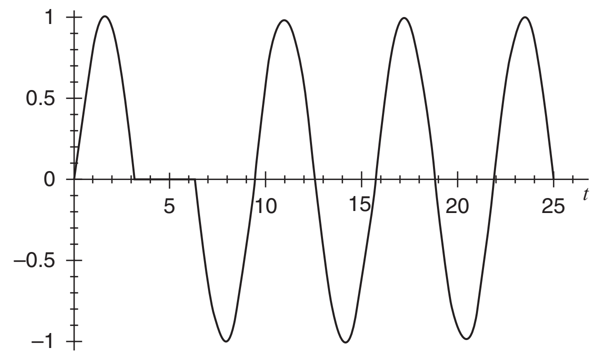
$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right) &= \sin(t-\pi)\cdot u(t-\pi)\\
&= -\sin t \cdot u(t-\pi) \qquad \text{(by periodicity of sine)}\\
\mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{s^2+1}\right) &= \sin(t-2\pi)\cdot u(t-2\pi)\\
&= \sin t \cdot u(t-2\pi)\\
\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) &= \sin t \qquad\qquad \text{(Table 6.1, Sec. 6.1).}
\end{aligned}$$

Together

$$y = -\sin t \cdot u(t-\pi) - \sin t \cdot u(t-2\pi) + \sin t.$$

Thus, from the effects of the unit step function,

$$y = \begin{cases} \sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \\ -\sin t & \text{if } t > 2\pi. \end{cases}$$



Sec. 6.4 Prob. 5. Solution curve $y(t)$

13(b). Heaviside formulas. The inverse transforms of the terms are obtained by the first shifting theorem,

$$\mathcal{L}^{-1}\left((s-a)^{-k}\right) = t^{k-1}e^{at} / (k-1)!.$$

Derive the coefficients as follows. Obtain A_m by multiplying the first formula in Prob. 13(b) by $(s-a)^m$, calling the new left side $Z(s)$:

$$Z(s) = (s-a)^m Y(s) = A_m + (s-a)A_{m-1} + \cdots + (s-a)^{m-1}A_1 + (s-a)^m W(s),$$

where $W(s)$ are the further terms resulting from other roots. Let $s \to a$ to get A_m,

$$A_m = \lim_{s \to a} [(s-a)^m Y(s)].$$

Differentiate $Z(s)$ to obtain

$$Z'(s) = A_{m-1} + \text{terms all containing factors } s-a.$$

Conclude that

$$Z'(a) = A_{m-1} + 0.$$

This is the coefficient formula with $k = m-1$, thus $m-k = 1$. Differentiate once more and let $s \to a$ to get

$$Z''(a) = 2!\,A_{m-2}$$

and so on.

Sec. 6.5 Convolution. Integral Equations

The sum $\mathcal{L}(f) + \mathcal{L}(g)$ is the transform $\mathcal{L}(f+g)$ of the sum $f+g$. However, the product $\mathcal{L}(f)\mathcal{L}(g)$ of two transforms is *not* the transform $\mathcal{L}(fg)$ of the product fg. What is it? It is the transform of the *convolution* $f * g$, whose defining integral is (see also p. 232)

$$\mathcal{L}(f)\mathcal{L}(g) = \mathcal{L}(f*g) \qquad \text{where} \qquad (f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau.$$

The main purpose of convolution is the solution of ODEs and the derivation of Laplace transforms, as the **Examples 1–5** (pp. 232–236) in the text illustrate. In addition, certain special classes of integral equations can also be solved by convolution, as in **Examples 6 and 7** on pp. 236–237.

Remark about solving integrals of convolution. When setting up the integral that defines convolution, keep the following facts in mind—they are an immediate consequence of the design of that integral.

1. When setting up the integrand, we replace t by τ in the first factor of the integrand and t by $t-\tau$ in the second factor, respectively.

2. Be aware that *we integrate with respect to* τ, not t!

3. Those factors of the integrand depending only on t are taken out from under the integral sign.

Problem 3, solved below, illustrates these facts in great detail.

Problem Set 6.5. Page 237

3. Calculation of convolution by integration. Result checked by Convolution Theorem.

Here

$$f(t) = e^t \qquad \text{and} \qquad g(t) = e^{-t}.$$

We need

$$f(\tau) = e^{\tau} \qquad \text{and} \qquad g(t-\tau) = e^{-(t-\tau)}, \tag{E1}$$

for determining the convolution. We compute the convolution $e^t * e^{-t}$ step by step.

The complete details with explanations are as follows.

$$\begin{aligned}
h(t) &= (f * g)(t) \\
&= e^t * e^{-t} \\
&= \int_0^t f(\tau) g(t-\tau)\, d\tau && \text{By definition of convolution on p. 232.} \\
&= \int_0^t e^{\tau} e^{-(t-\tau)}\, d\tau && \textbf{Attention}: \textit{integrate with respect to } \tau, \textit{ not } t! \\
&= \int_0^t e^{-t} e^{2\tau} d\tau && \text{Algebraic simplification of insertion from (E1).} \\
&= e^{-t} \int_0^t e^{2\tau}\, d\tau && \text{Take factor } e^{-t}\text{, depending only on } t\text{,} \\
& && \text{out from under the integral sign.} \\
&= \exp\left(-t \frac{e^{2\tau}}{2}\Bigg|_0^t\right) && \text{Integration.} \\
&= e^{-t}\left(\frac{e^{2t}}{2} - \frac{1}{2}\right) && \text{From evaluating bounds of integration.} \\
&= \tfrac{1}{2}(e^t - e^{-t}) && \text{Algebraic simplification.} \\
&= \sinh t && \text{By definition of } \sinh t.
\end{aligned}$$

Thus, the convolution of f and g is

$$(f * g)(t) = e^t * e^{-t} = \sinh t.$$

Checking the result by the Convolution Theorem on p. 232. Accordingly, we take the transform and verify that it is equal to $\mathcal{L}(e^t)\mathcal{L}(e^{-t})$. We have

$$\begin{aligned} H(s) &= \mathcal{L}(h)(t) \\ &= \mathcal{L}(\sinh t) \\ &= \frac{1}{s^2 - 1^2} \\ &= \frac{1}{s-1} \cdot \frac{1}{s+1} \\ &= \mathcal{L}(e^t)\mathcal{L}(e^{-t}) = \mathcal{L}(f)\mathcal{L}(g) = F(s)G(s), \end{aligned}$$

which is correct, by Theorem 1 (Convolution Theorem).

7. Convolution. We have $f(t) = t$, $g(t) = e^t$ and $f(\tau) = \tau$, $g(t - \tau) = e^{t-\tau}$. Then the convolution $f * g$ is

$$h(t) = (f * g)(t) = \int_0^t \tau e^{-(t-\tau)}\, d\tau = e^t \int_0^t \tau e^{-\tau}\, d\tau.$$

We use integration by parts ($u = \tau$, $v' = e^{-\tau}$) to first evaluate the indefinite integral and then evaluate the corresponding definite integral:

$$\int \tau e^{-\tau}\, d\tau = -\tau e^{-\tau} + \int e^{-\tau} d\tau = -\tau e^{-\tau} - e^{-\tau}; \qquad \int_0^t \tau e^{-\tau}\ d\tau = -te^{-t} - e^{-t} - 1.$$

Multiplying our result by e^t gives the convolution

$$h(t) = (f * g)(t) = e^t = e^t(-te^{-t} - e^{-t} - 1) = -t - 1 + e^t.$$

You may verify the answer by the Convolution Theorem as in Problem 3.

9. Integral equation. Looking at Examples 6 and 7 on pp. 236–237 and using (1) on p. 232 leads us to the following idea: The integral in $y(t) - \int_0^t y(\tau)\, d\tau = 1$ can be regarded as a convolution $1 * y$. Since 1 has the transform $1/s$, we can write the subsidiary equation as $Y - Y/s = 1/s$, thus $Y = 1/(s - 1)$ and obtain $y = e^t$.

Check this by differentiating the given equation, obtaining $y' - y = 0$, solve it, $y = ce^t$, and determine c by setting $t = 0$ in the given equation, $y(0) - 0 = 1$. (The integral is 0 for $t = 0$.)

25. Inverse transforms. We are given that

$$\mathcal{L}(e^t) = \frac{18s}{(s^2 + 36)^2}.$$

We want to find $f(t)$. We have to see how this fraction is put together by looking at Table 6.1 on p. 207 and also looking at Example 1 on p. 232. We notice that

$$\frac{18s}{(s^2 + 36)^2} = \frac{18s}{(s^2 + 6^2)^2} = \frac{18}{s^2 + 6^2} \cdot \frac{s}{s^2 + 6^2} = 3 \cdot \frac{6}{s^2 + 6^2} \cdot \frac{s}{s^2 + 6^2}.$$

The last equation is in a form suitable for direct application of the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(3 \cdot \frac{6}{s^2+6^2}\right) = 3 \cdot \mathcal{L}^{-1}\left(\frac{6}{s^2+6^2}\right) = 3\sin 6t, \qquad \mathcal{L}^{-1}\left(\frac{s}{s^2+6^2}\right) = \cos 6t.$$

Hence convolution gives you the inverse transform of

$$3 \cdot \frac{6}{s^2+6^2} \cdot \frac{s}{s^2+6^2} = \frac{18s}{(s^2+36)^2} \qquad \text{in the form} \qquad 3\sin 6t * \cos 6t.$$

We compute that convolution

$$3\sin 6t * \cos 6t = \int_0^t 3\sin 6\tau \cdot \cos 6(t-\tau)\, d\tau.$$

Now by formula (11) in Sec. A3.1 of App. 3 and simplifying

$$\sin 6\tau \cdot \cos 6(t-\tau) = \tfrac{1}{2}[\sin 6t + \sin(12\tau - 6t)]\,.$$

Hence the integral is

$$3\int_0^t \sin 6\tau \cdot \cos 6(t-\tau)\, d\tau = \frac{3}{2}\int_0^t (\sin 6t + \sin(12\tau - 6t))\ d\tau.$$

This breaks into two integrals. The first one evaluates to

$$\int_0^t \sin 6t\, d\tau = \sin 6t \int_0^t d\tau = t\sin 6t.$$

The second integral is, in indefinite form,

$$\int \sin(12\tau - 6t)\, d\tau = -\int \sin(6t - 12\tau)\, d\tau.$$

From calculus, substitution of $w = 6t - 12\tau$ (with $dw = -12\, d\tau$) yields

$$\int \sin(6t - 12\tau)\, d\tau = \frac{1}{12}\int \sin w\, dw = -\frac{\cos w}{12} = -\frac{1}{12}\cos(6t - 12\tau),$$

with the constant of integration set to 0. Hence the definite integral is

$$\begin{aligned}\int_0^t \sin(12\tau - 6t)\ d\tau &= \left[-\frac{1}{12}(\cos 6t - 12\tau)\right]_0^t \\ &= -\frac{1}{12}[\,\cos(6t - 12t) - \cos 6t] = -\frac{1}{12}[\,\cos(-6t) - \cos 6t] = 0.\end{aligned}$$

For the last step we used that cosine is an even function, so that $\cos(-6t) = \cos 6t$. Putting it all together,

$$3\sin 6t * \cos 6t = \tfrac{3}{2}(t\sin 6t + 0) = \tfrac{3}{2}t\sin 6t.$$

Sec. 6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

Do not confuse differentiation of *transforms* (Sec. 6.6) with differentiation of *functions* $f(t)$ (Sec. 6.2). The latter is basic to the whole transform method of solving ODEs. The present discussion on differentiation of transforms adds just another method of obtaining transforms and inverses. It completes some of the theory for Sec. 6.1 as shown on p. 238.

Also, solving ODEs with variable coefficients by the present method is restricted to a few such ODEs, of which the most important one is perhaps Laguerre's ODE (p. 240). This is because its solutions, the Laguerre polynomials, are orthogonal [by Team Project 14(b) on p. 504]. Our hard work has paid off and we have built such a large repertoire of techniques for dealing with Laplace transforms that we may have several ways of solving a problem. This is illustrated in the four solution methods in **Prob. 3**. The choice depends on what we notice about how the problem is put together, and there may be a preferred method as indicated in **Prob. 15.**

Problem Set 6.6. Page 241

3. Differentiation, shifting. We are given that $f(t) = \frac{1}{2}te^{-3t}$ and asked to find $\mathcal{L}(\frac{1}{2}te^{-3t})$. For better understanding we show that there are four ways to solve this problem.

Method 1. Use first shifting (Sec. 6.1). From Table 6.1, Sec. 6.1, we know that

$$\tfrac{1}{2}t \qquad \text{has the transform} \qquad \frac{\frac{1}{2}}{s^2}.$$

Now we apply the first shifting theorem (Theorem 2, p. 208) to conclude that

$$\left(\frac{1}{2}t\right)(e^{-3t}) \qquad \text{has the transform} \qquad \frac{\frac{1}{2}}{(s-(-3))^2}.$$

Method 2. Use differentiation, the preferred method of this section (Sec. 6.6). We have

$$\mathcal{L}(f(t)) = \mathcal{L}(e^{-3t}) = \frac{1}{s+3}$$

so that by (1), in the present section, we have

$$\mathcal{L}(tf) = \mathcal{L}\left(\frac{1}{2}te^{-3t}\right) = -\left(\frac{1}{2}\frac{1}{s+3}\right)' = -\left(-\frac{1}{2}\frac{1}{(s+3)^2}\right) = \frac{1}{2}\frac{1}{(s+3)^2} = \frac{\frac{1}{2}}{(s-(-3))^2}.$$

Method 3. Use of subsidiary equation (Sec. 6.2). As a third method, we write $g = \frac{1}{2}te^{-3t}$. Then $g(0) = 0$ and by calculus

$$g' = \tfrac{1}{2}e^{-3t} - 3\left(\tfrac{1}{2}te^{-3t}\right) = \tfrac{1}{2}e^{-3t} - 3g. \tag{A}$$

The subsidiary equation with $G = \mathcal{L}(g)$ is

$$sG = \frac{\frac{1}{2}}{s+3} - 3G, \qquad (s+3)G = \frac{\frac{1}{2}}{s+3}, \qquad G = \frac{\frac{1}{2}}{(s+3)^2}.$$

Method 4. Transform problem into second-order initial value problem (Sec. 6.2) and solve it. As a fourth method, an unnecessary detour, differentiate (A) to get a second-order ODE:

$$g'' = -\tfrac{3}{2}e^{-3t} - 3g' \qquad \text{with initial conditions} \qquad g(0) = 0,\ g'(0) = \tfrac{1}{2}$$

and solve the IVP by the Laplace transform, obtaining the same transform as before.

Note that the last two solutions concern IVPs involving different order ODEs, respectively.

15. Inverse transforms. We are given that

$$\mathcal{L}(f) = F(s) = \frac{s}{(s-9)^2}$$

and we want to find $f(t)$.
First solution. By straightforward differentiation we have

$$\frac{d}{ds}\left(\frac{1}{s^2-3^2}\right) = -\frac{2s}{\left(s^2-3^2\right)^2}.$$

Hence

$$\frac{s}{(s^2-3^2)^2} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2-3^2}\right).$$

Now by linearity of the inverse transform

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-3^2}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2-3^2}\right\} = \frac{1}{3}\sinh 3t,$$

where the last equality is obtained by Table 6.1, p. 207, by noting that $3/(s^2-3^2)$ is related to $\sinh 3t$. Putting it all together, and using formula (1) on p. 238,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{(s^2-3^2)^2}\right\} &= -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2-3^2}\right)\right\}\\ &= -\tfrac{1}{2}\left(-\tfrac{1}{3}t\sinh 3t\right)\\ &= \tfrac{1}{6}t\sinh 3t.\end{aligned}$$

Second solution. We can also solve this problem by convolution. Look back in this Student Solutions Manual at the answer to Prob. 25 of Sec. 6.5. We use the same approach:

$$\frac{s}{(s^2-9)^2} = \frac{s}{(s^2-3^2)^2} = \frac{1}{s^2-3^2}\cdot\frac{s}{s^2-3^2} = \frac{1}{3}\cdot\frac{3}{s^2-3^2}\cdot\frac{s}{s^2-3^2}.$$

This shows that the desired convolution is

$$\frac{1}{3}\sinh 3t * \cosh 3t = \int_0^t \frac{1}{3}(\sinh 3\tau)(\cosh 3(t-\tau))\,d\tau.$$

To solve this integral, use (17) of Sec. A3.1 in App. 3. This substitution will give four terms that involve only exponential functions. The integral breaks into four separate integrals that can be solved by calculus and, after substituting the limits of integration and using (17) again, would give us the same result as in the first solution. The moral of the story is that you can solve this problem in more than one way; however, the second solution is computationally more intense.

17. Integration of a transform. We note that

$$\lim_{T\to\infty}\left(\int_s^T \frac{d\widetilde{s}}{\widetilde{s}} - \int_s^T \frac{d\widetilde{s}}{\widetilde{s}-1}\right) = \lim_{T\to\infty}\left(\ln T - \ln s - \ln(T-1) + \ln(s-1)\right)$$

$$= \lim_{T\to\infty}\left(\ln\frac{T}{T-1} - \ln\frac{s}{s-1}\right) = -\ln\frac{s}{s-1}$$

so

$$\mathcal{L}^{-1}\left\{\ln\frac{s}{s-1}\right\} = -\mathcal{L}^{-1}\left\{\int_s^\infty \frac{d\widetilde{s}}{\widetilde{s}} - \int_s^\infty \frac{d\widetilde{s}}{\widetilde{s}-1}\right\} = -\left(\frac{1}{t} - \frac{e^t}{t}\right).$$

Sec. 6.7 Systems of ODEs

Note that the subsidiary system of a system of ODEs is obtained by the formulas of Theorem 1 in Sec. 6.2. This process is similar to that for a single ODE, except for notation. The section has beautiful applications: mixing problems (**Example 1**, pp. 242–243), electrical networks (**Example 2**, pp. 243–245), and springs (**Example 3**, pp. 245–246). Their solutions rely heavily on Cramer's rule and partial fractions. We demonstrate these two methods *in complete details* in our solutions to **Prob. 3** and **Prob. 15**, below.

Problem Set 6.7. Page 246

3. Homogeneous system. Use of the technique of partial fractions. We are given the following initial value problem stated as a homogeneous system of linear equations with two initial value conditions as follows:

$$\begin{aligned} y_1' &= -y_1 + 4y_2 \\ y_2' &= \ \ 3y_1 - 2y_2 \qquad \text{with} \qquad y_1(0) = 3, \qquad y_2(0) = 4. \end{aligned}$$

The subsidiary system is

$$\begin{aligned} sY_1 &= -Y_1 + 4Y_2 + 3, \\ sY_2 &= \ \ 3Y_1 - 2Y_2 + 4, \end{aligned}$$

where 3 and 4 on the right are the initial values of y_1 and y_2, respectively. This nonhomogeneous linear system can be written

$$\begin{aligned} (s+1)\,Y_1 + \quad -4Y_2 &= 3 \\ -3Y_1 + (s+2)Y_2 &= 4. \end{aligned}$$

We use **Cramer's rule** to solve the system. Note that Cramer's rule is useful for algebraic solutions but not for numerical work. This requires the following determinants:

$$D = \begin{vmatrix} s+1 & -4 \\ -3 & s+2 \end{vmatrix} = (s+1)(s+2) - (-4)(-3) = s^2 + 3s - 10 = (s-2)(s+5).$$

$$D_1 = \begin{vmatrix} 3 & -4 \\ 4 & s+2 \end{vmatrix} = 3(s+1) + 16 = 3s + 22.$$

$$D_2 = \begin{vmatrix} s+1 & 3 \\ -3 & 4 \end{vmatrix} = 4(s+1) - (3 \cdot (-3)) = 4s + 13.$$

Then

$$Y_1 = \frac{D_1}{D} = \frac{3s+22}{(s-2)(s+5)} \qquad Y_2 = \frac{D_2}{D} = \frac{4s+13}{(s-2)(s+5)}.$$

Next we use **partial fractions** on the results just obtained. We set up

$$\frac{3s+22}{(s-2)(s+5)} = \frac{A}{s-2} + \frac{B}{s+5}.$$

Multiplying the expression by $s-2$ and then substituting $s=2$ gives the value for A:

$$\frac{3s+22}{s+5} = A + \frac{B(s-2)}{s+5}, \qquad \frac{28}{7} = A + 0, \qquad \boxed{A = 4}.$$

Similarly, multiplying by $s-5$ and then substituting $s=5$ gives the value for B:

$$\frac{3s+22}{s-2} = \frac{A(s+5)}{s-2} + B, \qquad \frac{7}{-7} = 0 + B, \qquad \boxed{B = -1}.$$

This gives our first setup for applying the Laplace transform:

$$Y_1 = \frac{3s+22}{(s-2)(s+5)} = \frac{4}{s-2} + \frac{-1}{s+5}.$$

For the second partial fraction we have

$$\frac{4s+13}{(s-2)(s+5)} = \frac{C}{s-2} + \frac{D}{s+5},$$

$$\frac{4s+13}{s+5} = C + \frac{D(s-2)}{s+5}, \qquad \frac{21}{7} = C + 0, \qquad \boxed{C = 3}.$$

$$\frac{4s+13}{s-2} = \frac{C(s+5)}{s-2} + D, \qquad \frac{-7}{-7} = 0 + D, \qquad \boxed{D = 1}.$$

$$Y_2 = \frac{4s+13}{(s-2)(s+5)} = \frac{3}{s-2} + \frac{1}{s+5}.$$

Using Table 6.1, in Sec. 6.1, we obtain our final solution:

$$y_1 = \mathcal{L}^{-1}\left(\frac{4}{s-2}\right) + \mathcal{L}^{-1}\left(\frac{-1}{s+5}\right) = 4\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = 4e^{2t} - e^{-5t}.$$

$$y_2 = \mathcal{L}^{-1}\left(\frac{3}{s-2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = 3\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = 3e^{2t} + e^{-5t}.$$

15. Nonhomogeneous system of three linear ODEs. Use of the technique of convolution. The given initial value problem is

$$
\begin{aligned}
&y_1' + y_2' = 2\sinh t,\\
&y_2' + y_3' = e^t,\\
&y_3' + y_1' = 2e^t + e^{-t}, \qquad \text{with } y_1(0) = 1,\ y_2(0) = 1,\ y_3(0) = 0.
\end{aligned}
$$

Step 1. Obtain its subsidiary system with the initial values inserted:

$$
\begin{aligned}
(sY_1 - 1) + (sY_2 - 1) \qquad &= 2\frac{1}{s^2-1},\\
(sY_2 - 1) + sY_3 &= \frac{1}{s-1},\\
(sY_1 - 1) \qquad\qquad + sY_3 &= 2\frac{1}{s-1} + \frac{1}{s+1}.
\end{aligned}
$$

Simplification and written in a more convenient form gives

$$
\begin{aligned}
sY_1 + sY_2 \qquad &= \frac{2}{s^2-1} + 2,\\
sY_2 + sY_3 &= \frac{1}{s-1} + 1,\\
sY_1 \qquad + sY_3 &= \frac{2}{s-1} + \frac{1}{s+1} + 1.
\end{aligned}
$$

Step 2. Solve the auxiliary system by, say, Cramer's rule.

$$
D = \begin{vmatrix} s & s & 0\\ 0 & s & s\\ s & 0 & s \end{vmatrix} = s\cdot\begin{vmatrix} s & s\\ 0 & s\end{vmatrix} + s\cdot\begin{vmatrix} s & s\\ s & s\end{vmatrix} = s\cdot s^2 + s\cdot s^2 = 2s^3,
$$

$$
\begin{aligned}
D_1 &= \begin{vmatrix} \dfrac{2}{s^2-1} + 2 & s & 0\\ \dfrac{1}{s-1} + 1 & s & s\\ \dfrac{2}{s-1} + \dfrac{1}{s+1} + 1 & 0 & s \end{vmatrix} = -s\begin{vmatrix} \dfrac{2}{s^2-1} + 2 & s\\ \dfrac{2}{s-1} + \dfrac{1}{s+1} + 1 & 0 \end{vmatrix} + s\begin{vmatrix} \dfrac{2}{s^2-1} + 2 & s\\ \dfrac{1}{s-1} + 1 & s \end{vmatrix}\\
&= -s\left[-s\left(\frac{2}{s-1} + \frac{1}{s+1} + 1\right)\right] + s\left[s\left(\frac{2}{s^2-1} + 2\right) - s\left(\frac{1}{s-1} + 1\right)\right]\\
&= \frac{s^2}{s-1} + \frac{s^2}{s+1} + \frac{2s^2}{s^2-1} + 2s^2.
\end{aligned}
$$

$$D_2 = \begin{vmatrix} s & \dfrac{2}{s^2-1}+2 & 0 \\ 0 & \dfrac{1}{s-1}+1 & s \\ s & \dfrac{2}{s-1}+\dfrac{1}{s+1}+1 & s \end{vmatrix} = s\begin{vmatrix} \dfrac{1}{s-1}+1 & s \\ \dfrac{2}{s-1}+\dfrac{1}{s+1}+1 & s \end{vmatrix} + s\begin{vmatrix} \dfrac{2}{s^2-1}+2 & 0 \\ \dfrac{1}{s-1}+1 & s \end{vmatrix}$$

$$= s\left[s\left(\frac{1}{s-1}+1\right) - s\left(\frac{2}{s-1}+\frac{1}{s+1}+1\right)\right] - s\left[s\left(\frac{2}{s^2-1}+2\right)\right]$$

$$= -\frac{s^2}{s-1} - \frac{s^2}{s+1} + \frac{2s^2}{s^2-1} + 2s^2.$$

$$D_3 = \begin{vmatrix} s & s & \dfrac{2}{s^2-1}+2 \\ 0 & s & \dfrac{1}{s-1}+1 \\ s & s & \dfrac{2}{s-1}+\dfrac{1}{s+1}+1 \end{vmatrix} = s\begin{vmatrix} s & \dfrac{1}{s-1}+1 \\ 0 & \dfrac{2}{s-1}+\dfrac{1}{s+1}+1 \end{vmatrix} + s\begin{vmatrix} s & \dfrac{2}{s^2-1}+2 \\ s & \dfrac{1}{s-1}+1 \end{vmatrix}$$

$$= s\left[s\left(\frac{2}{s-1}+\frac{1}{s+1}+1\right)\right] + s\left[s\left(\frac{1}{s-1}+1\right) - s\left(\frac{2}{s^2-1}+2\right)\right]$$

$$= \frac{3s^2}{s-1} + \frac{s^2}{s+1} - \frac{2s^2}{s^2-1}.$$

Now that we have determinants D, D_1, D_2, and D_3 we get the solution to the auxiliary system by forming ratios of these determinants and simplifying:

$$Y_1 = \frac{D_1}{D} = \frac{1}{2s^3}\left[\frac{s^2}{s-1} + \frac{s^2}{s+1} + \frac{2s^2}{s^2-1} + 2s^2\right]$$

$$= \frac{1}{2}\cdot\frac{1}{s(s-1)} + \frac{1}{2}\cdot\frac{1}{s(s+1)} + \frac{1}{s(s^2-1)} - s^2.$$

$$Y_2 = \frac{D_2}{D} = \frac{1}{2s^3}\left[-\frac{s^2}{s-1} - \frac{s^2}{s+1} + \frac{2s^2}{s^2-1} + 2s^2\right]$$

$$= -\frac{1}{2}\cdot\frac{1}{s(s-1)} - \frac{1}{2}\cdot\frac{1}{s(s+1)} + \frac{1}{s(s^2-1)} + \frac{1}{s}.$$

$$Y_3 = \frac{D_3}{D} = \frac{1}{2s^3}\left[\frac{3s^2}{s-1} + \frac{s^2}{s+1} - \frac{2s^2}{s^2-1}\right]$$

$$= \frac{3}{2}\cdot\frac{1}{s(s-1)} + \frac{1}{2}\cdot\frac{1}{s(s+1)} - \frac{1}{s(s^2-1)}.$$

Step 3. Using Table 6.1, Sec. 6.1, potentially involving such techniques as linearity, partial fractions, convolution, and others from our toolbox of Laplace techniques, present Y_1, Y_2, and Y_3 in a form suitable for direct application of the inverse Laplace transform.

For this particular problem, it turns out that the "building blocks" of Y_1, Y_2, and Y_3 consist of linear combinations of

$$\frac{1}{s(s-1)} \qquad \frac{1}{s(s+1)} \qquad \frac{1}{s(s^2-1)}$$

to which we can apply the technique of convolution. Indeed, looking at Example 1, on p. 232 in Sec. 6.5, gives us most of the ideas toward the solution:

$$\frac{1}{s(s-1)} = \frac{1}{s-1}\cdot\frac{1}{s} \qquad \text{corresponds to convolution:} \qquad e^t * 1 = \int_0^t e^\tau\, d\tau = e^t - 1.$$

$$\frac{1}{s(s+1)} = \frac{1}{s+1}\cdot\frac{1}{s} \qquad \text{corresponds to convolution:} \qquad e^{-t} * 1 = \int_0^t e^{-\tau}\, d\tau = -(e^{-t} - 1).$$

$$\frac{1}{s(s^2-1)} = \frac{1}{s^2-1}\cdot\frac{1}{s} \qquad \text{corresponds to convolution:} \qquad \sinh t * 1 = \int_0^t \sinh\tau\, d\tau = \cosh t - 1.$$

Step 4. Put together the final answer:

$$\begin{aligned}
y_1 = \mathcal{L}^{-1}(Y_1) &= \tfrac{1}{2}(e^t - 1) + \tfrac{1}{2}(-e^{-t} + 1) + (\cosh t - 1) + 1 \\
&= \tfrac{1}{2}e^t - \tfrac{1}{2} - \tfrac{1}{2}e^{-t} + \tfrac{1}{2} + (\cosh t - 1) + 1 \\
&= \sinh t + \cosh t \\
&= e^t \qquad \text{[by formula (19), p. A65 in Sec. A3.1 of App. 3].}
\end{aligned}$$

$$\begin{aligned}
y_2 = \mathcal{L}^{-1}(Y_2) &= -\tfrac{1}{2}(e^t - 1) - \tfrac{1}{2}(-(e^{-t} - 1)) + (\cosh t - 1) + 1 \\
&= -\tfrac{1}{2}e^t + \tfrac{1}{2} + \tfrac{1}{2}e^{-t} - \tfrac{1}{2} + \cosh t \\
&= -\tfrac{1}{2}e^t + \tfrac{1}{2}e^{-t} + \cosh t \\
&= -\sinh t + \cosh t, \qquad \text{since } -\sinh t = -\tfrac{1}{2}e^t + \tfrac{1}{2}e^{-t} \\
&= e^{-t} \qquad \text{by formula (19), p. A65.}
\end{aligned}$$

$$\begin{aligned}
y_3 = \mathcal{L}^{-1}(Y_3) &= \tfrac{3}{2}(e^t - 1) + \tfrac{1}{2}(-(e^{-t} - 1)) - (\cosh t - 1) \\
&= \tfrac{3}{2}e^t - \tfrac{3}{2} - \tfrac{1}{2}e^{-t} + \tfrac{1}{2} - \cosh t + 1 \\
&= \tfrac{3}{2}e^t - \tfrac{1}{2}e^{-t} - \cosh t \\
&= \tfrac{3}{2}e^t - \tfrac{1}{2}e^{-t} - \tfrac{1}{2}e^t - \tfrac{1}{2}e^{-t} \qquad \text{(writing out } \cosh t\text{)} \\
&= e^t - e^{-t}.
\end{aligned}$$

PART B

Linear Algebra. Vector Calculus

Chap. 7 Linear Algebra: Matrices, Vectors, Determinants. Linear Systems

Although you may have had a course in linear algebra, we start with the basics. A matrix (Sec. 7.1) is an array of numbers (or functions). While most matrix operations are straightforward, **matrix multiplication** (Sec. 7.2) is nonintuitive. The heart of Chap. 7 is Sec. 7.3, which covers the famous **Gauss elimination method**. We use matrices to represent and solve systems of linear equations by Gauss elimination with back substitution. This leads directly to the theoretical foundations of linear algebra in Secs. 7.4 and 7.5 and such concepts as rank of a matrix, linear independence, and basis. A variant of the Gauss method, called Gauss–Jordan, applies to computing the **inverse of matrices** in Sec. 7.8.

The material in this chapter—with enough practice of solving systems of linear equations by Gauss elimination—should be quite manageable. Most of the theoretical concepts can be understood by thinking of practical examples from Sec. 7.3. Getting a good understanding of this chapter will help you in Chaps. 8, 20, and 21.

Sec. 7.1 Matrices, Vectors: Addition and Scalar Multiplication

Be aware that we can only **add** two (or more) matrices *of the **same** dimension*. Since addition proceeds by adding corresponding terms (see **Example 4**, p. 260, or **Prob. 11** below), the restriction on the dimension makes sense, because, if we would attempt to add matrices of different dimensions, we would run out of entries. For example, for the matrices in **Probs. 8–16**, we cannot add matrix **C** to matrix **A**, nor calculate $\mathbf{D} + \mathbf{C} + \mathbf{A}$.

Example 5, p. 260, and **Prob. 11** show scalar multiplication.

Problem Set 7.1. Page 261

11. Matrix addition, scalar multiplication. We calculate $8\mathbf{C} + 10\mathbf{D}$ as follows. First, we multiply the given matrix $\mathbf{C}$ by 8. This is an example of *scalar multiplication*. We have

$$8\mathbf{C} = 8\begin{bmatrix} 5 & 2 \\ -2 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 8\cdot 5 & 8\cdot 2 \\ 8\cdot(-2) & 8\cdot 4 \\ 8\cdot 1 & 8\cdot 0 \end{bmatrix} = \begin{bmatrix} 40 & 16 \\ -16 & 32 \\ 8 & 0 \end{bmatrix}.$$

Then we compute $10\mathbf{D}$ and get

$$10\mathbf{D} = 10\begin{bmatrix} -4 & 1 \\ 5 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -40 & 10 \\ 50 & 0 \\ 20 & -10 \end{bmatrix}.$$

The resulting matrices have the same size as the given ones, namely 3×2 (3 rows, 2 columns) because scalar multiplication does not alter the size of a matrix. Hence the operations of addition and subtraction are defined for these matrices, and we obtain the result by adding the entries of $10\mathbf{D}$ to the corresponding ones of $8\mathbf{C}$, that is,

$$\begin{aligned} 8\mathbf{C} + 10\mathbf{D} &= \begin{bmatrix} 40 & 16 \\ -16 & 32 \\ 8 & 0 \end{bmatrix} + \begin{bmatrix} -40 & 10 \\ 50 & 0 \\ 20 & -10 \end{bmatrix} \\ &= \begin{bmatrix} 40+(-40) & 16+10 \\ -16+50 & 32+0 \\ 8+20 & 0+(-10) \end{bmatrix} = \begin{bmatrix} 0 & 26 \\ 34 & 32 \\ 28 & -10 \end{bmatrix}. \end{aligned}$$

The next task is to calculate $2(5\mathbf{D} + 4\mathbf{C})$. We expect to obtain the same result as before. Why? Since $\mathbf{D}$ and $\mathbf{C}$ have the same dimensions, we can apply the rules for matrix addition and scalar multiplication as given on pp. 260–261 and get

$$\begin{aligned} 2(5\mathbf{D} + 4\mathbf{C}) &= (2\cdot 5)\mathbf{D} + (2\cdot 4)\mathbf{C} && \text{by rule (4c) applied twice} \\ &= 10\mathbf{D} + 8\mathbf{C} && \text{basic algebra for scalars (which are just numbers)} \\ &= 8\mathbf{C} + 10\mathbf{D} && \text{by commutativity of matrix addition, rule (3a).} \end{aligned}$$

Having given a general abstract algebraic derivation (which is a formal proof!) as to why we expect the same result, you should verify directly $2(5\mathbf{D} + 4\mathbf{C}) = 8\mathbf{C} + 10\mathbf{D}$ by doing the actual computation. You should calculate $5\mathbf{D}$, $4\mathbf{C}$, $5\mathbf{D} + 4\mathbf{C}$ and finally $2(5\mathbf{D} + 4\mathbf{C})$ and compare.

The next task is to compute $0.6\mathbf{C} - 0.6\mathbf{D}$. Thus, from the definition of matrix addition and scalar multiplication by -1 (using that $0.6\mathbf{C} - 0.6\mathbf{D} = 0.6\mathbf{C} + (-1)(0.6\mathbf{D})$), we have termwise subtraction:

$$0.6\mathbf{C} - 0.6\mathbf{D} = \begin{bmatrix} 3.0 & 1.2 \\ -1.2 & 2.4 \\ 0.6 & 0 \end{bmatrix} - \begin{bmatrix} -2.4 & 0.6 \\ 3.0 & 0 \\ 1.2 & -0.6 \end{bmatrix} = \begin{bmatrix} 5.4 & 0.6 \\ -4.2 & 2.4 \\ -0.6 & 0.6 \end{bmatrix}.$$

Computing $0.6(\mathbf{C} - \mathbf{D})$ gives the same answer. Show this, using similar reasoning as before.

15. Vectors are special matrices, having a single row or a single column, and operations with them are the same as for general matrices and involve fewer calculations. The vectors **u, v**, and **w** are column vectors, and they have the same number of components. They are of the same size 3×1. Hence they can be added to each other. We have

$$(\mathbf{u}+\mathbf{v})-\mathbf{w} = \left(\begin{bmatrix} 1.5 \\ 0 \\ -3.0 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}\right) - \begin{bmatrix} -5 \\ -30 \\ 10 \end{bmatrix} = \begin{bmatrix} 1.5-1 \\ 0+3 \\ -3.0+2 \end{bmatrix} - \begin{bmatrix} -5 \\ -30 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 3.0 \\ -1.0 \end{bmatrix} - \begin{bmatrix} -5 \\ -30 \\ 10 \end{bmatrix} = \begin{bmatrix} 0.5-(-5) \\ 3-(-30) \\ -1.0-10 \end{bmatrix} = \begin{bmatrix} 5.5 \\ 33.0 \\ -11.0 \end{bmatrix}.$$

Next we note that $\mathbf{u}+(\mathbf{v}-\mathbf{w}) = (\mathbf{u}+\mathbf{v})-\mathbf{w}$, since

$$\begin{aligned}
\mathbf{u}+(\mathbf{v}-\mathbf{w}) &= \mathbf{u}+(\mathbf{v}+(-1)\mathbf{w}) && \text{by definition of vector subtraction as addition and scalar multiplication by } -1. \\
&= (\mathbf{u}+\mathbf{v})+(-1)\mathbf{w} && \text{by rule (3b).} \\
&= (\mathbf{u}+\mathbf{v})-\mathbf{w} && \text{by scalar multiplication.} \\
&= \mathbf{u}+(\mathbf{v}-\mathbf{w}) && \text{by associativity rule 3(b).}
\end{aligned}$$

We see that $\mathbf{C}+0\mathbf{w}$ is undefined, since $\mathbf{C}$ is a 3×2 matrix and

$$0\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is a 3×1 matrix (or column vector of dimension 3) and thus cannot be added to $\mathbf{C}$, since the dimensions of the two matrices are not the same. Also $0\mathbf{E}+\mathbf{u}-\mathbf{v}$ is undefined.

19. Proof of (3a). **A** and **B** are assumed to be general 2×3 matrices. Hence let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}.$$

Now by the definition of matrix addition, we obtain

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

and

$$\mathbf{B}+\mathbf{A} = \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} & b_{13}+a_{13} \\ b_{21}+a_{21} & b_{22}+a_{22} & b_{23}+a_{23} \end{bmatrix}.$$

Now remember what you want to prove. You want to prove that $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$. By definition, you want to prove that corresponding entries on both sides of this matrix equation are equal. Hence you want to prove that

(a) $$a_{11}+b_{11} = b_{11}+a_{11}$$

and similarly for the other five pairs of entries. Now comes the *idea*, the key of the proof. Use the commutativity of the addition of *numbers* to conclude that the two sides of (a) are equal. Similarly for the other five pairs of entries. This completes the proof.

The proofs of all the other formulas in (3) of p. 260 and (4) of p. 261 follow the same pattern and the same idea. Perform all these proofs to make sure that you really understand the logic of our procedure of proving such general matrix formulas. In each case, the equality of matrices follows from the corresponding property for the equality of numbers.

Sec. 7.2 Matrix Multiplication

The key concept that you have to understand in this section is **matrix multiplication.** Take a look at its definition and **Example 1** on p. 263. Matrix multiplication proceeds by "row *times* column." Your left index finger can sweep horizontally along the rows of the first matrix, and your right index finger can sweep vertically along the columns of the matrix. You proceed "first row" times "first column" forming the sum of products, then "first row" times "second column" again forming the sums of products, etc. For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 4 & 6 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 10 & 11 & 20 & 50 \\ 40 & 30 & 80 & 60 \end{bmatrix}$$

then **AB** (that is, matrix **A** multiplied by matrix **B**) is [*please close this Student Solutions Manual (!) and see if you can do it by paper and pencil or type on your computer without looking and then compare the result*]:

$$\mathbf{AB} = \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 10 & 11 & 20 & 50 \\ 40 & 30 & 80 & 60 \end{bmatrix}.$$

(See p. 128 of this chapter for the solution.)

If you got the correct answer, great. If not, see where you went wrong and try again. Our main point is that you have to **memorize how to do matrix multiplication** *because in the exam it is most unlikely that you will be able to derive matrix multiplication*! Although matrix multiplication is not particularly difficult, it is not intuitive. Furthermore, note that in our example, **BA** is undefined, as the number of rows of **B** (4 rows) does not equal the number of columns of **A** (2 columns)! (If you try to multiply **BA** you run out of entries.)

Distinguish between scalar multiplication (Sec. 7.1) and matrix multiplication (Sec. 7.2). *Scalar multiplication*, in the present context, means the multiplication of a matrix (or vector) by a scalar (a real number). *Matrix multiplication* means the multiplication of two (or more) matrices, including vectors as special cases of matrices.

Matrix multiplication is not commutative, that is, in general $\mathbf{AB} \neq \mathbf{BA}$, as shown in **Example 4** on p. 264. [This is different from basic multiplication of numbers, that is, $19 \cdot 38 = 38 \cdot 19 = 342$, or factors involving variables $(x+3)(x+9) = (x+9)(x+3) = x^2 + 12x + 27$, or terms of an ODE $(x^2 - x)y'' = y''(x^2 - x)$, etc.] Thus matrix multiplication forces us to carefully distinguish between matrix multiplication *from the left*, as in (2d) on p. 264 where $(\mathbf{A} + \mathbf{B})$ is multiplied by matrix **C** *from the left,* resulting in $\mathbf{C}(\mathbf{A} + \mathbf{B})$ versus matrix multiplication *from the right* in (2c) resulting in $(\mathbf{A} + \mathbf{B})\mathbf{C}$.

The "strange" definition of matrix multiplication is initially motivated on pp. 265–266 (linear transformations) and *more fully motivated* on pp. 316–317 (composition of linear transformations, **new!**).

You should remember special classes of square matrices (symmetric, skew-symmentric, triangular, diagonal, and scalar matrices) introduced on pp. 267–268 as they will be needed quite frequently.

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11. Matrix multiplication, transpose, symmetric matrix. We note that the condition for matrices **A** and **B**

$$\text{Number of columns of the first factor} = \text{Number of rows of the second factor}$$

is satisfied: **A** has 3 columns and **B** has 3 rows. Let's compute **AB** by "row times column" (as explained in the text, in our example above, and in an illustration on p. 263)

$$\mathbf{AB} = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \cdot 1 + (-2)(-3) + 3 \cdot 0 & 4 \cdot (-3) + (-2) \cdot 1 + 3 \cdot 0 & 4 \cdot 0 + (-2) \cdot 0 + 3 \cdot (-2) \\ (-2) \cdot 1 + 1 \cdot (-3) + 6 \cdot 0 & (-2)(-3) + 1 \cdot 1 + 6 \cdot 0 & (-2) \cdot 0 + 1 \cdot 0 + 6 \cdot (-2) \\ 1 \cdot 1 + 2 \cdot (-3) + 2 \cdot 0 & 1 \cdot (-3) + 2 \cdot 1 + 2 \cdot 0 & 1 \cdot 0 + 2 \cdot 0 + 2 \cdot (-2) \end{bmatrix}$$

$$= \begin{bmatrix} 4 + 6 + 0 & -12 - 2 + 0 & 0 + 0 - 6 \\ -2 - 3 + 0 & 6 + 1 + 0 & 0 + 0 - 12 \\ 1 - 6 + 0 & -3 + 2 + 0 & 0 + 0 - 4 \end{bmatrix} = \begin{bmatrix} 10 & -14 & -6 \\ -5 & 7 & -12 \\ -5 & -1 & -4 \end{bmatrix}. \tag{M1}$$

To obtain the transpose $\mathbf{B}^\mathsf{T}$ of a matrix **B**, we write the rows of **B** as columns, as explained on pp. 266–267 and in Example 7. Here $\mathbf{B}^\mathsf{T} = \mathbf{B}$, since **B** is a special matrix, that is, a *symmetric* matrix:

$$b_{21} = b_{12} = -3, \qquad b_{31} = b_{13} = 0, \qquad b_{32} = b_{23} = 0.$$

And, as always for any square matrix, the elements on the main diagonal (here $b_{11} = 4$, $b_{22} = 1$, $b_{33} = 2$) are not affected, since their subscripts are

$$b_{jj} \quad j = 1, 2, 3.$$

In general $\mathbf{B} \neq \mathbf{B}^\mathsf{T}$. We have for our particular problem that $\mathbf{AB}^\mathsf{T} = \mathbf{AB}$ and we get the same result as in (M1).

$$\mathbf{BA} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 + 6 + 0 & -2 - 3 + 0 & 3 - 18 + 0 \\ -12 - 2 + 0 & 6 + 1 + 0 & -9 + 6 + 0 \\ 0 + 0 - 2 & 0 + 0 - 4 & 0 + 0 - 4 \end{bmatrix} = \begin{bmatrix} 10 & -5 & -15 \\ -14 & 7 & -3 \\ -2 & -4 & -4 \end{bmatrix}. \tag{M2}$$

This shows that $\mathbf{AB} \neq \mathbf{BA}$, *which is true in general*! Furthermore, here $\mathbf{B}^\mathsf{T}\mathbf{A} = \mathbf{BA}$, which is equal to (M2).

15. Multiplication of a matrix by a vector will be needed in connection with linear systems of equations, beginning in Sec. 7.3. **Aa** is *undefined* since matrix **A** has 3 columns but vector **a** has only 1 row (and we know that the number of columns of **A** must be equal to the number of rows of **a** for the multiplication to be defined). However, the condition for allowing matrix multiplication is satisfied for $\mathbf{Aa}^{\mathsf{T}}$:

$$\mathbf{Aa}^{\mathsf{T}} = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 + (-2)(-2) + 3 \cdot 0 \\ (-2) \cdot 1 - 1 \cdot (-2) + 6 \cdot 0 \\ 1 \cdot 1 + 2 \cdot (-2) + 2 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 + 4 + 0 \\ -2 - 2 + 0 \\ 1 - 4 + 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}.$$

Similarly, calculate $\mathbf{Ab}$ and get

$$\mathbf{Ab} = \begin{bmatrix} 7 \\ -11 \\ 3 \end{bmatrix} \qquad \text{so that} \qquad (\mathbf{Ab})^{\mathsf{T}} = \begin{bmatrix} 7 \\ -11 \\ 3 \end{bmatrix}^{\mathsf{T}} = [7 \quad -11 \quad 3].$$

Since the product $(\mathbf{Ab})^{\mathsf{T}}$ is defined, we have, by (10d) on p. 267, that $(\mathbf{Ab})^{\mathsf{T}} = \mathbf{b}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} = [7 \quad -11 \quad 3]$. Note that (10d) holds for any appropriately dimensioned matrices $\mathbf{A}$, $\mathbf{B}$ and thus also applies to matrices and vectors.

21. General rules. Proceed as in Prob. 19 of Sec. 7.1, as explained in this Manual. In particular, to show (2c) that $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$ we start as follows. We let $\mathbf{A}$ be as defined in (3) on p. 125 of Sec. 4.1. Similarly for $\mathbf{B}$ and $\mathbf{C}$. Then, by matrix addition, p. 260,

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{11} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{11} \\ c_{21} & c_{22} \end{bmatrix}.$$

By matrix multiplication, the first entry, which we write out by distributivity of numbers, is as below.

$$\text{(E1)} \qquad (a_{11} + b_{11})c_{11} + (a_{12} + b_{11})c_{21} = a_{11}c_{11} + b_{11}c_{11} + a_{12}c_{21} + b_{11}c_{21}.$$

Similarly for the other entries of $(\mathbf{A} + \mathbf{B})\mathbf{C}$. For the right-hand side of (2c) you set up $\mathbf{AC} + \mathbf{BC}$. The first entry we obtain can then be rearranged by *commutativity of numbers*:

$$\text{(E2)} \qquad a_{11}c_{11} + a_{12}c_{21} + b_{11}c_{11} + b_{11}c_{21} = a_{11}c_{11} + b_{11}c_{11} + a_{12}c_{21} + b_{11}c_{21}.$$

But this shows that the right-hand side of (E2) is precisely equal to the right-hand side of (E1), which we obtained for $(\mathbf{A} + \mathbf{B})\mathbf{C}$. After writing out these steps for all other three entries, you have proven (2c) on p. 264. By using the same approach as outlined for (2c), you will be able to prove (2a), (2b), and (2d). Always remember that these proofs hinge on the fact that individual entries in these matrices are numbers and, as such, obey the rules with which you are familiar.

29. Application: profit vector. If we denote by p_S the profit per sofa, by p_C the profit per chair, and by p_T the profit per table, then we can denote by $\mathbf{p} = [p_S \quad p_C \quad p_T]^{\mathsf{T}}$ the profit vector. To compute the total profit per week for F_1 and F_2, respectively, we need

$$\mathbf{v} = \mathbf{Ap} = \begin{bmatrix} 400 & 60 & 240 \\ 100 & 120 & 500 \end{bmatrix} \begin{bmatrix} p_S \\ p_C \\ p_T \end{bmatrix} = \begin{bmatrix} 400p_S + 60p_C + 240p_T \\ 100p_S + 120p_C + 500p_T \end{bmatrix}.$$

We are given that

$$p_S = \$85, \qquad p_C = \$62, \qquad p_T = \$30,$$

so that

$$\mathbf{v} = \mathbf{A}\mathbf{p} = \begin{bmatrix} 400p_S + \ \ 60p_C + 240p_T \\ 100p_S + 120p_C + 500p_T \end{bmatrix} = \begin{bmatrix} 400 \cdot \$85 + \ \ 60 \cdot \$62 + 240 \cdot \$30 \\ 100 \cdot \$85 + 120 \cdot \$62 + 500 \cdot \$30 \end{bmatrix}.$$

This simplifies to $[\$44{,}920 \qquad \$30{,}940]^{\mathsf{T}}$ as given on p. A17.

Sec. 7.3 Linear Systems of Equations. Gauss Elimination

This is the heart of Chap. 7. Take a careful look at **Example 2** on pp. 275–276. First you do **Gauss elimination**. This involves changing the augmented matrix $\widetilde{\mathbf{A}}$ (also preferably denoted by $[\mathbf{A} \mid \mathbf{b}]$ on p. 279) to an upper triangular matrix (4) by elementary row operations. They are (p. 277) interchange of two equations (rows), addition of a constant multiple of one equation (row) to another equation (row), and multiplication of an equation (a row) by a *nonzero* constant c. The method involves the strategy of "eliminating" ("reducing to 0") all entries in the augmented matrix that are below the main diagonal. You obtain matrix (4). Then you do **back substitution**. **Problem 3** of this section gives another carefully explained example.

A system of linear equations can have a *unique solution* (**Example 2**, pp. 275–276), *infinitely many solutions* (**Example 3**, p. 278), and *no solution* (**Example 4**, pp. 278–279).

Look at **Example 4**. No solution arises because Gauss gives us "$0 = 12$," that is, "$0x_3 = 12$," which is impossible to solve. The equations have no point in common. The geometric meaning is parallel planes (Fig. 158) or, in two dimensions, parallel lines. You need to practice the important technique of Gauss elimination and back substitution. Indeed, Sec. 7.3 serves as the background for the theory of Secs. 7.4 and 7.5. Gauss elimination appears in many variants, such as in computing inverses of matrices (Sec. 7.8, called Gauss–Jordan method) and in solving elliptic PDEs numerically (Sec. 21.4, called Liebmann's method).

Problem Set 7.3. Page 280

3. Gauss elimination. Unique solution. *Step 1. Construct the augmented matrix.* We express the given system of three linear nonhomogeneous equations in the form $\mathbf{Ax} = \mathbf{b}$ as follows:

$$\begin{aligned} x + \ y - \ z &= \ 9 \\ 8y + 6z &= -6 \\ -2x + 4y - 6z &= \ 40 \end{aligned} \qquad \text{which is} \qquad \begin{bmatrix} 1 & 1 & -1 \\ 0 & 8 & 6 \\ -2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 40 \end{bmatrix}.$$

From this we build the corresponding augmented matrix $[\mathbf{A} \mid \mathbf{b}]$:

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \end{array}\right] \qquad \text{is of the general form} \qquad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right].$$

The first three columns of $[\mathbf{A} \mid \mathbf{b}]$ are the coefficients of the equations and the fourth column is the right-hand side of these equations. Thus, the matrix summarizes the information without any loss of information.

Step 2. Perform the Gauss elimination on the augmented matrix. It is, of course, perfectly acceptable to do the Gauss elimination in terms of equations rather than in terms of the augmented matrix.

However, as soon as you feel sufficiently acquainted with matrices, you may wish to save work in writing by operating on matrices. The unknowns are eliminated in the order in which they occur in each equation. Hence start with x. Since x occurs in the first equation, we **pivot** the first equation. *Pivoting means that we mark an equation and use it to eliminate entries in the equations below. More precisely, we pivot an entry and use this entry to eliminate (get rid of) the entries directly below*. It is practical to indicate the operations after the corresponding row, as shown in Example 2 on p. 276 of the textbook. You obtain the next matrix row by row. Copy Row 1. This is the pivot row in the first step. Variable x does not occur in Eq. (2), so we need not operate on Row 2 and simply copy it. To eliminate x from Eq. (3), we add 2 times Row 1 to Row 3. Mark this after Row 3 of the following matrix $[\widetilde{\mathbf{A}} \mid \widetilde{\mathbf{b}}]$, which, accordingly, takes the form

$$[\widetilde{\mathbf{A}} \mid \widetilde{\mathbf{b}}] = \left[\begin{array}{rrr|r} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 6 & -8 & 58 \end{array}\right] \begin{array}{l} \\ \\ \text{Row } 3 + 2\,\text{Row } 1. \end{array}$$

If you look it over, nothing needed to be done for a_{21}, since a_{21} was already zero. We removed $a_{31} = -2$ with the pivot $a_{11} = 1$, and knew that $a_{31} + 2a_{11} = 0$ as desired. This determined the algebraic operation of Row $3 + 2$ Row 1. Note well that rows may be left unlabeled if you do not operate on them. And the row numbers occurring in labels always refer to the *previous matrix* just as in the book. Variable x has now been eliminated from all but the first row. We turn to the next unknown, y. We copy the first two rows of the present matrix and operate on Row 3 by subtracting from it $\frac{3}{4}$ times Row 2 because this will eliminate y from Eq. (3). Thus, the new pivot row is Row 2. The result is

$$[\mathbf{R} \mid \mathbf{f}] = \left[\begin{array}{rrr|r} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 0 & -\frac{25}{2} & \frac{125}{2} \end{array}\right] \begin{array}{l} \\ \\ \text{Row } 3 - \frac{3}{4}\,\text{Row } 2. \end{array}$$

If you look over the steps going from $[\widetilde{\mathbf{A}} \mid \widetilde{\mathbf{b}}]$ to $[\mathbf{R} \mid \mathbf{f}]$, understand the following. *To get the* ***correct*** *number of times we wanted to subtract Row 2 from Row 3,* we noted that $\widetilde{a}_{22} = 8$ and $\widetilde{a}_{32} = 6$. Hence we need to find a value of k such that $\widetilde{a}_{32} - k\widetilde{a}_{22} = 0$. This is, of course, $k = \widetilde{a}_{32}/\widetilde{a}_{22} = \frac{6}{8} = \frac{3}{4}$. Thus, we had the operation Row $3 - \frac{3}{4}$ Row 2, so that for the new entry r_{32} in matrix $[\mathbf{R} \mid \mathbf{f}]$

$$r_{32} = \widetilde{a}_{32} - \tfrac{3}{4}\widetilde{a}_{22} = 6 - \tfrac{3}{4} \cdot 8 = 6 - 6 = 0.$$

Further computations for r_{33} and f_3 were

$$r_{33} = \widetilde{a}_{33} - \tfrac{3}{4}\widetilde{a}_{23} = -8 - \tfrac{3}{4} \cdot 6 = -\tfrac{25}{2}; \qquad f_3 = \widetilde{b}_3 - \tfrac{3}{4}\widetilde{b}_2 = 58 - \tfrac{3}{4} \cdot (-6) = \tfrac{125}{2}.$$

There was no computation necessary for r_{31} by design of the elimination process. Together the last row of $[\mathbf{R} \mid \mathbf{f}]$ became $[r_{31} \quad r_{32} \quad r_{33} \quad f_3] = [0 \quad 0 \ -\frac{25}{2} \quad \frac{125}{2}]$. We have now reached the end of the Gauss elimination process, since the matrix obtained, denoted by $[\mathbf{R} \mid \mathbf{f}]$, is in (upper) triangular form.

Step 3. Do the back substitution to obtain the final answer. We write out the three equations

$$\begin{array}{lrcl} (1) & x + y - z &=& 9, \\ (2) & 8y + 6z &=& -6, \\ (3) & -\frac{25}{2}z &=& \frac{125}{2}. \end{array}$$

Equation (3) gives

$$z = \tfrac{125}{2} \cdot \left(-\tfrac{2}{25}\right) = -5, \qquad \boxed{z = -5}.$$

Substituting $z = -5$ into Eq. (2) gives us

$$y = \tfrac{1}{8}(-6 - 6z) = \tfrac{1}{8}(-6 + (-6)(-5)) = \tfrac{1}{8}(-6 + 30) = \tfrac{24}{8} = 3 \qquad \boxed{y = 3}.$$

Substituting $y = 3$ and $z = -5$ into Eq. (3) yields

$$x = 9 - y + z = 9 - 3 + (-5) = 9 - 3 - 5 = 9 - 8 = 1 \qquad \boxed{x = 1}.$$

Thus, we obtained the unique solution

$$x = 1, \qquad y = 3, \qquad z = -5.$$

This is possibility (b) on p. 280 for solutions of linear systems of equations and illustrated by Example 2, pp. 275–276 in the textbook.

Remark. In the back substitution process, when doing the problem by hand, it may be easier to substitute the value(s) obtained into the equations directly, simplify and solve, instead of first writing down the equation with the wanted variable isolated on the left-hand side and the other variables on the right-hand side and then substituting the values (as we did here for conceptual clarity). Thus, the alternative approach, suggested here, would be to substitute the result from Eq. (3), that is, $z = -5$ into Eq. (1) directly:

$$8y + 6z = 8y + 6(-5) = 8y - 30 = -6; \qquad 8y = -6 + 30 = 24; \qquad y = \tfrac{24}{8} = 3.$$

Furthermore,

$$x + y - z = x + 3 - (-5) = x + 3 + 5 = x + 8 = 9; \qquad x = 9 - 8 = 1.$$

Step 4. Check your answer by substituting the result back into the original linear system of equations.

$$\begin{aligned}
x + y - z &= 1 + 3 - (-5) &&= 1 + 3 + 5 &&= 9. \checkmark \\
8y + 6z &= 8 \cdot 3 + 6 \cdot (-5) &&= 24 - 30 &&= -6. \checkmark \\
-2x + 4y - 6z &= (-2) \cdot 1 + 4 \cdot 3 - 6(5) &&= -2 + 12 + 20 &&= 40. \checkmark
\end{aligned}$$

Our answer is correct, because $9, -6, 40$ are the right-hand sides of the original equations.

7. Gauss elimination. Infinitely many solutions. From the given linear homogeneous system we get the augmented matrix

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{rrr|r} 2 & 4 & 1 & 0 \\ -1 & 1 & -2 & 0 \\ 4 & 0 & 6 & 0 \end{array}\right].$$

Using Row 1 as the pivot row, we eliminate $a_{21} = -1$ and $a_{31} = 4$ by means of the pivot $a_{11} = 2$ and get $[\widetilde{\mathbf{A}} \mid \widetilde{\mathbf{b}}]$. Then we have to pivot Row 2 to get rid of the remaining off-diagonal entry -8 in Row 3. We get $[\mathbf{R} \mid \mathbf{f}]$.

$$[\widetilde{\mathbf{A}} \mid \widetilde{\mathbf{b}}] = \left[\begin{array}{rrr|r} 2 & 4 & 1 & 0 \\ 0 & 3 & -\frac{3}{2} & 0 \\ 0 & -8 & 4 & 0 \end{array}\right] \begin{array}{l} \\ \text{R2} + \frac{1}{2}\,\text{R1} \\ \text{R3} - 2\,\text{R1}. \end{array} \qquad [\mathbf{R} \mid \mathbf{f}] = \left[\begin{array}{rrr|r} 2 & 4 & 1 & 0 \\ 0 & 3 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{R3} + \frac{8}{3}\,\text{R2}. \end{array}$$

If you look back at how we got $[\mathbf{R} \mid \mathbf{f}]$, to get the correct number of times we want to add Row 2 (denoted by "R2") to Row 3 ("R3"), we note that $\widetilde{a}_{22} = 3$ and $\widetilde{a}_{32} = -8$. We find the value of k, such that $\widetilde{a}_{32} - k\widetilde{a}_{22} = 0$, that is, $k = \widetilde{a}_{32}/\widetilde{a}_{22} = -\frac{8}{3}$. Since this value was *negative* we did a row *addition*. For the back substitution, we write out the system

$$2x + 4y + \quad z = 0. \tag{4}$$

$$3y - \tfrac{3}{2}z = 0. \tag{5}$$

The last row of $[\mathbf{R} \mid \mathbf{f}]$ consists of all zeroes and thus does not need to be written out. (It would be $0 = 0$). Equation (5) gives

$$z = 3 \cdot \tfrac{2}{3}y = 2y, \qquad \boxed{z = 2y}.$$

Substituting this into Eq. (4) gives

$$x = \tfrac{1}{2}(-4y - z) = \tfrac{1}{2}(-4y - 2y) = -3y \qquad \boxed{x = -3y}.$$

Furthermore,

$$\boxed{y \text{ is arbitrary.}}$$

Thus, we have one parameter; let us call it t. We set $y = t$. Then the final solution becomes (see p. A17)

$$x = -3y = -3t, \qquad y = t, \qquad z = 2y = 2t.$$

Remark. You could have solved Eq. (5) for y and obtained

$$\boxed{y = \tfrac{1}{2}z}$$

and substituted that into Eq. (4) to get

$$x = \tfrac{1}{2}\big((-4) \cdot \big(\tfrac{1}{2}z\big) - z\big) = \tfrac{1}{2}(-2z - z) = -\tfrac{3}{2}z \qquad \boxed{x = -\tfrac{3}{2}z}.$$

Now

$$\boxed{z \text{ is arbitrary.}}$$

Then you would set $z = t$ and get

$$x = -\tfrac{3}{2}z = -\tfrac{3}{2}t, \qquad y = \tfrac{1}{2}z = \tfrac{1}{2}t, \qquad z = t,$$

which is also a correct answer. (Finally, but less likely, you could also have chosen x to be arbitrary and obtained the result $x = t$, $y = -\frac{1}{3}t$, and $z = -\frac{2}{3}t$.) The moral of the story is that we can choose which variable we set to the parameter t with *one* choice allowed in this problem. This example illustrates possibility (c) on p. 280 in the textbook, that is, infinitely many solutions (as we can choose any value for the parameter t and thus have infinitely many choices for t) and Example 3 on p. 278.

17. Electrical network. We are given the elements of the circuits, which, in this problem, are batteries and Ohm's resistors. The first step is the introduction of letters and directions for the unknown currents, which we want to determine. This has already been done in the figure of the network as shown. We do not know the directions of the currents. However, this does not matter. We make a choice, and if an unknown current comes out negative, this means that we have chosen the wrong direction and the current actually flows in the opposite direction. There are three currents I_1, I_2, I_3; hence we need three equations. An obvious choice is the right node, at which I_3 flows in and I_1 and I_2 flow out; thus, by KCL (Kirchhoff's Current Law, see Sec. 2.9 (pp. 93–99) and also Example 2, pp. 275–276),

$$I_3 = I_1 + I_2.$$

The left node would do equally well. Can you see that you would get the same equation (except for a minus sign by which all three currents would now be multiplied)? Two further equations are obtained from KVL (Kirchhoff's Voltage Law, Sec. 2.9), one for the upper circuit and one for the lower one. In the upper circuit, we have a voltage drop of $2I_1$ across the right resistor. Hence the sum of the voltage drops is $2I_1 + I_3 + 2I_1 = 4I_1 + I_3$. By KVL this sum equals the electromotive force 16 on the upper circuit; here resistance is measured in ohms and voltage in volts. Thus, the second equation for determining the currents is

$$4I_1 + I_3 = 16.$$

A third equation is obtained from the lower circuit by KVL. The voltage drop across the left resistor is $4I_2$ because the resistor has resistance of 4 Ω and the current I_2 is flowing through it, causing a drop. A second voltage drop occurs across the upper (horizontal) resistor in the circuit, namely $1 \cdot I_3$, as before. The sum of these two voltage drops must equal the electromotive force of 32 V in this circuit, again by KVL. This gives us

$$4I_2 + I_3 = 32.$$

Hence the system of the three equations for the three unknowns, properly ordered, is

$$\begin{aligned} I_1 + \ \ I_2 - I_3 &= \ \ 0. \\ 4I_1 \qquad\ + I_3 &= 16. \\ 4I_2 - I_3 &= 32. \end{aligned}$$

From this, we immediately obtain the corresponding augmented matrix:

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 4 & 0 & 1 & 16 \\ 0 & 4 & 1 & 32 \end{array}\right] \quad \text{is of the form} \quad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right].$$

The pivot row is Row 1 and the pivot $a_{11} = 1$. Subtract 4 times Row 1 from Row 2, obtaining

$$[\widetilde{\mathbf{A}} \mid \widetilde{\mathbf{b}}] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -4 & 5 & 16 \\ 0 & 4 & 1 & 32 \end{array}\right] \text{Row 2} - 4 \text{ Row 1}.$$

Note that Row 1, which is the pivot row (or pivot equation), remains untouched. Since now both $\widetilde{a}_{21} = 0$ and $\widetilde{a}_{31} = 0$, we need a new pivot row. The new pivot row is Row 2. We use it to eliminate $\widetilde{a}_{32} = 4$, which corresponds to I_2 (having a coefficient of 4) from Row 3. To do this we add Row 2 to Row 3, obtaining

$$[\mathbf{R} \,|\, \mathbf{f}] = \left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & -4 & 5 & 16 \\ 0 & 0 & 6 & 48 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3 + Row 2.} \end{array}$$

Now the system has reached triangular form, that is, all entries below the main diagonal of $\mathbf{R}$ are 0. This means that $\mathbf{R}$ is in row echelon form (p. 279) and the Gauss elimination is done. Now comes the back substitution. First, let us write the transformed system in terms of equations from

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \\ 48 \end{bmatrix}$$

and obtain

$$\begin{aligned} I_1 + \;\; I_2 - \;\; I_3 &= \;\; 0. \\ -4I_2 + 5I_3 &= 16. \\ 6I_3 &= 48. \end{aligned}$$

From Eq. (3) we obtain

$$6I_3 = 48; \qquad I_3 = \tfrac{48}{6} = 8.$$

We substitute this into Eq. (2) and get

$$-4I_2 + 5I_3 = -4I_2 + 5 \cdot 8 = 16; \quad -4I_2 + 40 = 16; \quad -4I_2 = 16 - 40 = -24; \quad I_2 = \tfrac{-24}{-4} = 6.$$

Finally, by substituting $I_2 = 6$ and $I_3 = 8$ into Eq. (1), we get

$$I_1 + I_2 - I_3 = 0; \qquad I_1 = I_3 - I_1 = 8 - 6 = 2.$$

Thus, the final answer is $I_1 = 2$ [A] (amperes), $I_2 = 6$ [A], and $I_3 = 8$ [A].

Sec. 7.4 Linear Independence. Rank of a Matrix. Vector Space

Linear independence and dependence is of general interest throughout linear algebra. Rank will be the central concept in our further discussion of existence and uniqueness of solutions of linear systems in Sec. 7.5.

Problem Set 7.4. Page 287

1. Rank by inspection. The first row equals -2 times the second row, that is, $[4 \quad -2 \quad 6] = -2[-2 \quad 1 \quad 3]$. Hence the rank of the matrix is at most 1. It cannot be 0 because the given matrix does not contain all zeros as entries. Hence the rank of the matrix $= 1$. The first column equals -2 times the second column; furthermore, the first column equals $\frac{2}{3}$ times the third column:

$$\begin{bmatrix} 4 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}; \qquad \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 6 \\ -3 \end{bmatrix}.$$

From these two relationships together, we conclude that the rank of the transposed matrix (and hence the matrix) is 1. Another way to see this is to reduce

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \\ 6 & -3 \end{bmatrix} \quad \text{to} \quad \begin{array}{l} \begin{bmatrix} 4 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Row 2} + \frac{1}{2}\text{ Row 1} \\ \text{Row 3} - \frac{3}{2}\text{ Row 1} \end{array}$$

which clearly has rank 1. A basis for the row space is $[4 \quad -2 \quad 6]$ (or equivalently $[2 \quad -1 \quad 3]$ obtained by division of 2, as given on p. A17). A basis for the column space is $[4 \quad -2]^{\mathsf{T}}$ or equivalently $[2 \quad -1]^{\mathsf{T}}$. *Remark.* In general, if v_1, v_2, v_3 form a basis for a vector space, so do c_1v_1, c_2v_2, c_3v_3 for any constants c_1, c_2, c_3 *all* different from 0. Hence any nonzero multiple of $[4 \quad -2 \quad 6]$ and any nonzero multiple of $[4 \quad -2]^{\mathsf{T}}$ are valid answers for a basis. The row basis and the column basis here consists of only *one* vector, respectively, as the rank is *one*.

3. Rank by row reduction. In the given matrix, since the first row starts with a zero entry and the second row starts with a nonzero entry, we take the given matrix and interchange Row 1 and Row 2.

$$\begin{bmatrix} 0 & 3 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 10 \end{bmatrix} \qquad \begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 5 & 0 & 10 \end{bmatrix} \begin{array}{l} \text{Row 2} \\ \text{Row 1.} \\ \end{array}$$

Then the "new" Row 1 becomes the pivot row and we calculate Row 3 $-\ \frac{5}{3}$ Row 1. Next Row 2 becomes the pivot row and we calculate Row 3 $-\ \frac{1}{3} \cdot \frac{25}{3}$ Row 2. The two steps are

$$\begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 0 & -\frac{25}{3} & 10 \end{bmatrix} \begin{array}{l} \\ \\ \text{Row 3} - \frac{5}{3}\text{ Row 1;} \end{array} \qquad \begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{215}{9} \end{bmatrix} \begin{array}{l} \\ \\ \text{Row 3} - \frac{1}{3}\cdot\frac{25}{3}\text{ Row 2.} \end{array}$$

The matrix is in row-reduced form and has 3 nonzero rows. Hence the rank of the given matrix is 3. Since the given matrix is symmetric (recall definition, see pp. 267–268) the transpose of the given matrix is the same as the given matrix. Hence the rank of the transposed matrix is 3. A basis for the row space is $[3 \quad 5 \quad 0]$, $[0 \quad 3 \quad 5]$, and $[0 \quad 0 \quad 1]$ (last row multiplied by $9/215$). By transposition, a basis for the column space is $[3 \quad 5 \quad 0]^{\mathsf{T}}$, $[0 \quad 3 \quad 5]^{\mathsf{T}}$, $[0 \quad 0 \quad 1]^{\mathsf{T}}$.

13. Rank of square. A counterexample is as follows. rank $\mathbf{A} =$ rank $\mathbf{B} = 1$:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

but rank $(\mathbf{A}^2) = 0 \neq$ rank $(\mathbf{B}^2) = 1$ because

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{B}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

21. Linear dependence. We form a matrix with the given vectors $[2 \quad 0 \quad 0 \quad 7]$, $[2 \quad 0 \quad 0 \quad 8]$, $[2 \quad 0 \quad 0 \quad 9]$, $[2 \quad 0 \quad 1 \quad 0]$ as rows and interchange Row 4 with Row 1; pivot the "new" Row 1 and do the three row reductions as indicated:

$$\begin{bmatrix} 2 & 0 & 0 & 7 \\ 2 & 0 & 0 & 8 \\ 2 & 0 & 0 & 9 \\ 2 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 8 \\ 2 & 0 & 0 & 9 \\ 2 & 0 & 0 & 7 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -8 \\ 0 & 0 & -1 & -9 \\ 0 & 0 & -1 & -7 \end{bmatrix} \begin{array}{l} \\ \text{Row 2} - \text{Row 1} \\ \text{Row 3} - \text{Row 1} \\ \text{Row 4} - \text{Row 1.} \end{array}$$

Then we pivot Row 2 and do two row reductions; pivot Row 3 and add Row 3 to Row 4:

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -8 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{array}{l} \\ \\ \text{Row 3} - \text{Row 2} \\ \text{Row 4} - \text{Row 2;} \end{array} \qquad \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -8 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \text{Row 4} + \text{Row 3.} \end{array}$$

Since the last row is 0, the matrix constructed from the 4 vectors does not have the full rank of 4 but has rank 3. Thus, the four given vectors are **linearly dependent** by Theorem 4, p. 285, with $p = 4$, $n = 4$, and $rank = 3 < p = 4$.

Sec. 7.5 Solution of Linear Systems: Existence, Uniqueness

Remember the main fact that a linear system of equations has solutions if and only if the coefficient matrix and the augmented matrix have the same rank. See **Theorem 1** on p. 288.

Hence a *homogeneous* linear system always has the trivial solution $\mathbf{x} = \mathbf{0}$. It has nontrivial solutions if the rank of its coefficient matrix is less than the number of unknowns.

The dimension of the solution space equals the number of unknowns minus the rank of the coefficient matrix. Hence the smaller that rank is, the "more" solutions will the system have. In our notation [see (5) on p. 291]

$$\text{nullity}\,\mathbf{A} = n - \text{rank}\,\mathbf{A}.$$

Sec. 7.6 For Reference: Second- and Third-Order Determinants

Cramer's rule for systems in 2 and 3 unknowns, shown in this section, is obtained by elimination of unknowns. Direct elimination (e.g., by Gauss) is generally simpler than the use of Cramer's rule.

Sec. 7.7 Determinants. Cramer's Rule

This section explains how to calculate determinants (pp. 293–295) and explores their properties (**Theorem 2**, p. 297). Cramer's rule is given by **Theorem 4**, p. 298, and applied in **Prob. 23** below. Note that the significance of determinants has decreased (as larger matrices are needed), certainly in computations, as can be inferred from the table in **Prob. 4** on p. 300.

Problem Set 7.7. Page 300

7. Evaluate determinant.

$$\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix} = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta) \qquad \text{[by (6), p. A64 in App. 3].}$$

15. Evaluation of determinants. We use the method of Example 3, p. 295, and Example 4, p. 296. Employing the idea that the determinant of a row-reduced matrix is related to the determinant of the original matrix by Theorem 1, p. 295, we first reduce the given matrix using Row 1 as the pivot row:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 16 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 16 \end{bmatrix} \begin{matrix} \\ \text{Row 2} - 2 \text{ Row 1.} \\ \\ \\ \end{matrix}$$

We interchange Row 2 and Row 3 and then use Row 3 as the pivot row:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 16 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix} \begin{matrix} \\ \\ \\ \text{Row 4} - \text{Row 3.} \end{matrix}$$

Since this matrix is in triangular form, we can immediately figure out the determinant of this reduced matrix. We have det(reduced matrix) $= (-1) \cdot$ det(original matrix), with the multiplicative factor of -1 due to one row interchange (!) by Theorem 1, part (a), p. 295. Thus, we obtain

$$\begin{aligned} \det(\text{original matrix}) &= -1 \cdot \det(\text{reduced matrix}) \\ &= -1 \cdot (\text{product of the entries in the diagonal}) \\ &= -1 \cdot (1 \cdot 2 \cdot 2 \cdot 16) = -64. \end{aligned}$$

23. Cramer's rule. The given system can be written in the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is the 3×3 matrix of the coefficients of the variables, $\mathbf{x}$ a vector of these variables, and $\mathbf{b}$ the vector corresponding to the right-hand side of the given system. Thus, we have

$$\begin{aligned} 3y - 4z &= 16 \\ 2x - 5y + 7z &= -27 \\ -x \qquad - 9z &= 9 \end{aligned} \qquad \text{which is} \qquad \begin{bmatrix} 0 & 3 & -4 \\ 2 & -5 & 7 \\ -1 & 0 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -27 \\ 9 \end{bmatrix}.$$

Applying Theorem 4, p. 298, to our system of three nonhomogeneous linear equations, we proceed as follows. Note that we can develop the determinants along any column or row. The signs of the cofactors are determined by the following checkerboard pattern:

$$\begin{matrix} + & - & + \\ - & + & -. \\ + & - & + \end{matrix}$$

Note that we try to develop along columns or rows that have the largest number of zero entries. This simplifies our hand calculations. The determinant D of the system is

$$D = \det \mathbf{A} = \begin{vmatrix} 0 & 3 & -4 \\ 2 & -5 & 7 \\ -1 & 0 & -9 \end{vmatrix} = 0 \cdot \begin{vmatrix} -5 & 7 \\ 0 & -9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & -4 \\ 0 & -9 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 3 & -4 \\ -5 & 7 \end{vmatrix}$$

$$= (-2)(3 \cdot (-9) - (-4) \cdot 0) - (3 \cdot 7 - (-4)(-5)) = (-2)(-27) - (21 - 20) = 54 - 1 = 53.$$

The computation concerning the three determinants in the numerators of the quotients for the three unknowns comes next. Note that we obtain D_1 by taking D and replacing its first column by the entries of vector **b**, D_2 by replacing the second column of D with **b**, and D_3 by replacing the third column.

For D_1 we develop along the second column (with the signs of the cofactors being $-, +, -$). Both determinants D_2 and D_3 are developed along the first column. Accordingly, the signs of the cofactors are $+, -, +$:

$$
\begin{aligned}
D_1 &= \begin{vmatrix} 16 & 3 & -4 \\ -27 & -5 & 7 \\ 9 & 0 & -9 \end{vmatrix} \\
&= (-3) \cdot \begin{vmatrix} -27 & 7 \\ 9 & -9 \end{vmatrix} + (-5) \cdot \begin{vmatrix} 16 & -4 \\ 9 & -9 \end{vmatrix} - 0 \cdot \begin{vmatrix} 16 & -4 \\ -27 & 7 \end{vmatrix} \\
&= (-3)((-27)(-9) - 7 \cdot 9) - 5 \cdot (16(-9) - (-4) \cdot 9) \\
&= (-3)(243 - 63) - 5 \cdot (-144 + 36) = (-3) \cdot 180 + 5 \cdot 108 = -540 + 540 = 0. \\
D_2 &= \begin{vmatrix} 0 & 16 & -4 \\ 2 & -27 & 7 \\ -1 & 9 & -9 \end{vmatrix} \\
&= 0 \cdot \begin{vmatrix} -27 & 7 \\ 9 & -9 \end{vmatrix} + 2 \cdot \begin{vmatrix} 16 & -4 \\ 9 & -9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 16 & -4 \\ -27 & 7 \end{vmatrix} \\
&= (-2)(16(-9) - (-4) \cdot 9) - 1 \cdot (16 \cdot 7 - (-4)(-27)) \\
&= (-2)(-144 + 36) - (112 - 108) = (-2)(-108) - 4 = 212. \\
D_3 &= \begin{vmatrix} 0 & 3 & 16 \\ 2 & -5 & -27 \\ -1 & 0 & 9 \end{vmatrix} = 0 \cdot \begin{vmatrix} 2 & -5 \\ 1 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 16 \\ 0 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 3 & 16 \\ -5 & -27 \end{vmatrix} \\
&= (-2) \cdot 27 - (-81 + 80) = -54 + 1 = -53.
\end{aligned}
$$

We obtain the values of the unknowns:

$$
x = \frac{D_1}{D} = \frac{0}{53} = 0, \qquad y = \frac{D_2}{D} = \frac{212}{53} = 4, \qquad z = \frac{D_3}{D} = \frac{-53}{53} = -1.
$$

Sec. 7.8 Inverse of a Matrix. Gauss–Jordan Elimination

The inverse of a square matrix **A** is obtained by the Gauss–Jordan elimination as explained in detail in **Example 1** on pp. 303–304 of the textbook. The example shows that the entries of the inverse will, in general, be fractions, even if the entries of **A** are integers.

The general formula (4) for the inverse (p. 304) is hardly needed in practice, whereas the special case (4*) is worth remembering.

Theorem 3 on p. 307 answers questions concerned with unusual properties of matrix multiplication. **Theorem 4** on the determinant of a product of matrices occurs from time to time in applications and theoretical derivations.

Problem Set 7.8. Page 308

3. Inverse. In the given matrix **A**, we can express the decimal entries in Row 1 as fractions, that is,

$$\mathbf{A} = \begin{bmatrix} 0.3 & -0.1 & 0.5 \\ 2 & 6 & 4 \\ 5 & 0 & 9 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} & \frac{1}{2} \\ 2 & 6 & 4 \\ 5 & 0 & 9 \end{bmatrix}.$$

Using **A**, we form

$$[\mathbf{A} \;\; \mathbf{I}] = \left[\begin{array}{ccc|ccc} \frac{3}{10} & -\frac{1}{10} & \frac{1}{2} & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 1 & 0 \\ 5 & 0 & 9 & 0 & 0 & 1 \end{array}\right],$$

which is our starting point. We follow precisely the approach of Example 1, pp. 303–304. The use of fractions, that is, writing 0.3 as $\frac{3}{10}$, $-0.1 = -\frac{1}{10}$, etc. gives a more accurate answer and minimizes rounding errors (see Chap. 19 on numerics). Go slowly and use a lot of paper to get the calculations with fractions right. We apply the *Gauss elimination* (Sec. 7.3) to the 3×6 matrix [**A I**]. Using Row 1 as the pivot row, eliminate 2 and 5. Calculate

$$\begin{array}{l} \left[\begin{array}{ccc|ccc} \frac{3}{10} & -\frac{1}{10} & \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{20}{3} & \frac{2}{3} & -\frac{20}{3} & 1 & 0 \\ 0 & \frac{5}{3} & \frac{2}{3} & -\frac{50}{3} & 0 & 1 \end{array}\right] \end{array} \begin{array}{ll} & \\ \text{Row } 2 - \frac{10}{3} \cdot 2 \text{ Row } 1 & \left(\text{i.e., Row } 2 - \frac{20}{3} \text{ Row } 1\right) \\ \text{Row } 3 - \frac{10}{3} \cdot 5 \text{ Row } 1 & \left(\text{i.e., Row } 3 - \frac{50}{3} \text{ Row } 1\right). \end{array}$$

Next we eliminate $\frac{5}{3}$ (the only entry left below the main diagonal), using Row 2 as the pivot row:

$$\left[\begin{array}{ccc|ccc} \frac{3}{10} & -\frac{1}{10} & \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{20}{3} & \frac{2}{3} & -\frac{20}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -15 & -\frac{1}{4} & 1 \end{array}\right] \begin{array}{ll} & \\ & \\ \text{Row } 3 - \frac{3}{20} \cdot \frac{5}{3} \text{ Row } 2 & \left(\text{i.e., Row } 3 - \frac{1}{4} \text{ Row } 2\right). \end{array}$$

This is [**U H**] obtained by Gauss elimination. Now comes the *Jordan part*, that is, the additional Gauss–Jordan steps, reducing the upper triangular matrix **U** to the identity matrix **I**, that is, to diagonal form with only entries 1 on the main diagonal. First, we multiply each row by appropriate numbers to obtain 1'*s* on the main diagonal:

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{5}{3} & \frac{10}{3} & 0 & 0 \\ 0 & 1 & \frac{1}{10} & -1 & \frac{3}{20} & 0 \\ 0 & 0 & 1 & -30 & -\frac{1}{2} & 2 \end{array}\right] \begin{array}{l} \frac{10}{3} \text{ Row } 1 \\ \frac{3}{20} \text{ Row } 2 \\ 2 \;\; \text{Row } 3. \end{array}$$

Using Row 3 as the pivot row, we eliminate the entries $\frac{5}{3}$ and $\frac{1}{10}$ above the main diagonal:

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & 0 & \frac{160}{3} & \frac{5}{6} & -\frac{10}{3} \\ 0 & 1 & 0 & 2 & \frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -30 & -\frac{1}{2} & 2 \end{array}\right] \begin{array}{l} \text{Row } 1 - \frac{5}{3} \text{ Row } 3 \\ \text{Row } 2 - \frac{1}{10} \text{ Row } 3. \\ \\ \end{array}$$

Finally, we eliminate $-\frac{1}{3}$ above the diagonal using Row 2 as the pivotal row:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 54 & \frac{9}{10} & -\frac{17}{5} \\ 0 & 1 & 0 & 2 & \frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -30 & -\frac{1}{2} & 2 \end{array}\right] \quad \text{Row 1} + \tfrac{1}{3}\,\text{Row 2.}$$

The system is in the form $[\mathbf{I}\ \mathbf{A}^{-1}]$, as explained on pp. 302–303. Thus, the desired inverse matrix is

$$\mathbf{A}^{-1} = \begin{bmatrix} 54 & \frac{9}{10} & -\frac{17}{5} \\ 2 & \frac{1}{5} & -\frac{1}{5} \\ -30 & -\frac{1}{2} & 2 \end{bmatrix} = \begin{bmatrix} 54 & 0.9 & -3.4 \\ 2 & 0.2 & -0.2 \\ -30 & -0.5 & 2 \end{bmatrix}.$$

Check the result in this problem by calculating the matrix products $\mathbf{AA}^{-1}$and $\mathbf{A}^{-1}\mathbf{A}$. Both should give you the matrix $\mathbf{I}$.

23. Formula (4) for the inverse. Probs. 21–23 should aid in understanding the use of minors and cofactors. The given matrix is

$$\mathbf{A} = \begin{bmatrix} 0.3 & -0.1 & 0.5 \\ 2 & 6 & 4 \\ 5 & 0 & 9 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} & \frac{1}{2} \\ 2 & 6 & 4 \\ 5 & 0 & 9 \end{bmatrix}.$$

In (4) you calculate $1/\det \mathbf{A} = 1/1 = 1$. Denote the inverse of $\mathbf{A}$ simply by $\mathbf{B} = [b_{jk}]$. Calculate the entries of (4):

$$b_{11} = c_{11} = \begin{vmatrix} 6 & 4 \\ 0 & 9 \end{vmatrix} = 54,$$

$$b_{12} = c_{21} = -\begin{vmatrix} -\frac{1}{10} & \frac{1}{2} \\ 0 & 9 \end{vmatrix} = -\left(-\frac{9}{10}\right) = \frac{9}{10},$$

$$b_{13} = c_{31} = \begin{vmatrix} -\frac{1}{10} & \frac{1}{2} \\ 6 & 4 \end{vmatrix} = -\frac{4}{10} - 3 = -\frac{4}{10} - \frac{30}{10} = -\frac{34}{10} = -\frac{17}{5},$$

$$b_{21} = c_{12} = -\begin{vmatrix} 2 & 4 \\ 5 & 9 \end{vmatrix} = -(18 - 20) = 2,$$

$$b_{22} = c_{22} = \begin{vmatrix} \frac{3}{10} & \frac{1}{2} \\ 5 & 9 \end{vmatrix} = \frac{27}{10} - \frac{5}{2} = \frac{27}{10} - \frac{25}{10} = \frac{2}{10} = \frac{1}{5},$$

$$b_{23} = c_{32} = -\begin{vmatrix} \frac{3}{10} & \frac{1}{2} \\ 2 & 4 \end{vmatrix} = -\left(\frac{12}{10} - 1\right) = -\left(\frac{12}{10} - \frac{10}{10}\right) = -\frac{2}{10} = -\frac{1}{5},$$

$$b_{31} = c_{13} = \begin{vmatrix} 2 & 6 \\ 5 & 0 \end{vmatrix} = 0 - 5 \cdot 6 = -30,$$

$$b_{32} = c_{23} = -\begin{vmatrix} \frac{3}{10} & -\frac{1}{10} \\ 5 & 0 \end{vmatrix} = -\left(-\left(-\frac{1}{10} \cdot 5\right)\right) = -\frac{5}{10} = -\frac{1}{2},$$

$$b_{33} = c_{33} = \begin{vmatrix} \frac{3}{10} & -\frac{1}{10} \\ 2 & 6 \end{vmatrix} = \frac{18}{10} + \frac{2}{10} = \frac{20}{10} = 2.$$

Putting it all together, we see that

$$\mathbf{A}^{-1} = \begin{bmatrix} 54 & 0.9 & -3.4 \\ 2 & 0.2 & -0.2 \\ -30 & -0.5 & 2 \end{bmatrix},$$

which is in agreement with our previous result in Prob. 3.

Sec. 7.9 Vector Spaces, Inner Product Spaces. Linear Transformations. *Optional*

The main concepts are **vector spaces** (pp. 309–310), **inner product spaces** (pp. 311–313), **linear transformations** (pp. 313–315), and their composition (new!) (pp. 316–317). The purpose of such concepts is to allow engineers and scientists to communicate in a concise and common language. It may take some time to get used to this more abstract thinking. It can be of help to think of practical examples underlying these abstractions.

Problem Set 7.9. Page 318

3. Vector space. We are given a set, call it S_3, consisting of all vectors in R^3 satisfying the linear system

$$-v_1 + 2v_2 + 3v_3 = 0, \tag{1}$$

$$-4v_1 + v_2 + v_3 = 0. \tag{2}$$

Solve the linear system and get

$$\begin{aligned} v_1 &= -\tfrac{1}{7}v_3, \\ v_2 &= -\tfrac{11}{7}v_3, \\ v_3 &\text{ is arbitrary.} \end{aligned}$$

Setting $v_3 = t$, we can write the solution as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7}t \\ -\frac{11}{7}t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{7} \\ -\frac{11}{7} \\ 1 \end{bmatrix} = t\mathbf{v}.$$

Thus, a basis for vector space S_3 (verification below that S_3 actually is a vector space) is $\mathbf{v}$ and also $-7\mathbf{v}$, that is, $[1 \quad 11 \quad -7]^{\mathsf{T}}$ is also a basis for S_3 (as on p. A18). We conclude that the dimension of S_3 is 1.

Now to show that S_3 *is actually* a vector space, let **a**, **b** be arbitrary elements of S_3. Since S_3 is a subset of R^3, we know that each of **a**, **b** has three real components. Then

$$\mathbf{a} = [a_1 \quad a_2 \quad a_3]^\mathsf{T} = [a_1 \quad 11a_1 \quad -7a_1]^\mathsf{T}, \tag{3}$$

$$\mathbf{b} = [b_1 \quad b_2 \quad b_3]^\mathsf{T} = [b_1 \quad 11b_1 \quad -7b_1]^\mathsf{T}, \tag{4}$$

so that their sum is

$$\mathbf{a} + \mathbf{b} = [a_1 + b_1 \quad 11a_1 + 11b_1 \quad -7a_1 - 7b_1]^\mathsf{T}. \tag{A}$$

To show that $\mathbf{a} + \mathbf{b}$ is S_3 we have to show that $\mathbf{a} + \mathbf{b}$ satisfies the original system. By substituting the components of $\mathbf{a} + \mathbf{b}$ into Eq. (1) we obtain

$$\begin{aligned} -v_1 + 2v_2 + 3v_3 &= -(a_1 + b_1) + 2(11a_1 + 11b_1) + 3(-7a_1 - 7b_1) \\ &= a_1(-1 + 22 - 21) + b_1(-1 + 22 - 21) \\ &= 0 \cdot a_1 + 0 \cdot a_2 = 0, \end{aligned}$$

which means that $\mathbf{a} + \mathbf{b}$ satisfies Eq. (1). The same holds true for Eq. (2) as you should show. This proves that $\mathbf{a} + \mathbf{b}$ is S_3. Next you show that **I.1** and **I.2** on p. 310 hold. The **0** vector is

$$\mathbf{0} = [0 \quad 0 \quad 0]^\mathsf{T} = a_1[1 \quad 11 \quad -7]^\mathsf{T}, \qquad \text{with } a_1 = 0.$$

It satisfies Eq. (1) and Eq. (2), since we know that zero is always a solution to a system of homogeneous linear equations. Furthermore, from computation with real numbers (each of the elements of the vector is a real number!),

$$\begin{aligned} \mathbf{a} + \mathbf{0} &= [a_1 \quad 11a_1 \quad -7a_1]^\mathsf{T} + [0 \quad 0 \quad 0]^\mathsf{T} \\ &= [a_1 + 0 \quad 11a_1 + 0 \quad -7a_1 + 0]^\mathsf{T} \\ &= [a_1 \quad 11a_1 \quad -7a_1]^\mathsf{T} = \mathbf{a}. \end{aligned}$$

Since these solution vectors to the system live in R^3 and we know that **0** is a solution to the system and, since **0** unique in R^3, we conclude **0** is a unique vector in S_3 being a subset of R^3. This shows that **I.3** holds for S_3. For **I.4** we need that

$$-\mathbf{a} = [-a_1 \quad -11a_1 \quad 7a_1]^\mathsf{T}.$$

It satisfies Eq. (1):

$$-(-a_1) + 2(-11a_1) + 21a_1 = a_1 - 22a_1 + 21a_1 = 0.$$

Similarly for the second equation. Furthermore,

$$\begin{aligned} \mathbf{a} + (-\mathbf{a}) &= [a_1 \quad 11a_1 \quad -7a_1]^\mathsf{T} + [-a_1 \quad -11a_1 \quad 7a_1]^\mathsf{T} \\ &= [a_1 + (-a_1) \quad 11a_1 + (-11a_1) \quad -7a_1 + 7a_1]^\mathsf{T} \\ &= [0 \quad 0 \quad 0]^\mathsf{T} = \mathbf{0}, \end{aligned}$$

which follows from (A) and the computation in R^3. Furthermore, for each component, the inverse is unique, so that together, the inverse vector is unique. This shows that **I.4** (p. 310) holds. Axioms **II.1,**

II.2, **III.3** are satisfied, as you should show. They hinge on the idea that if **a** satisfies Eq. (1) and Eq. (2), so does $k\mathbf{a}$ for any real scalar k. To show **II.4** we note that for

$$\begin{aligned} 1\mathbf{a} &= 1[a_1 \quad a_2 \quad a_3]^\mathsf{T} = 1[a_1 \quad 11a_1 \quad -7a_1]^\mathsf{T} \\ &= [1a_1 \quad 1 \cdot 11a_1 \quad 1 \cdot (-7)a_1]^\mathsf{T} = \mathbf{a}, \end{aligned}$$

so that **II.4** is satisfied. After you have filled in all the indicated missing steps, you get a complete proof that S_3 is indeed a vector space.

5. Not a vector space. From the problem description, we consider a set, call it S_5, consisting of polynomials of the form (under the usual addition and scalar multiplication of polynomials)

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \tag{B}$$

with the added condition

$$a_0, a_1, a_2, a_3, a_4 \geqq 0. \tag{C}$$

S_5 is not a vector space. The problem lies in condition (C), which violates Axiom **I.4.** Indeed, choose some coefficients not all zero, say, $a_0 = 1$ and $a_4 = 7$ (with the others zero). We obtain a polynomial $1 + 7x^4$. It is clearly in S_5. Its inverse is

$$-(1 + 7x^4) = -1 - 7x^4 \tag{A1}$$

because $(1 + 7x^4) + (-1 - 7x^4) = 0$. However, (A1) is not in S_5 as its coefficients -1 and -7 are negative, violating condition (C). Conclude that S_5 is not a vector space. Note the strategy: If we can show that a set S (with given addition and scalar multiplication) violates *just one* of the axioms on p. 310, then S (under the given operations) is not a vector space. Can you find another polynomial whose inverse is not in S_5?

11. Linear transformation. In vector form we have $\mathbf{y} = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 0.5 & -0.5 \\ 1.5 & -2.5 \end{bmatrix}.$$

The inverse is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$. Hence **Probs. 11–14** are solved by determining the inverse of the coefficient matrix **A** of the given transformation (if it exists, that is, if **A** is nonsingular). We use the method of Sec. 7.8, that is,

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \qquad \text{[by (4*), p. 304].}$$

We have $\det \mathbf{A} = (0.5) \cdot (-2.5) - (-0.5)(1.5) = -1.25 + 0.75 = -0.5$. Thus,

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{-0.5} \begin{bmatrix} -2.5 & 0.5 \\ -1.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 5.0 & -1.0 \\ 3.0 & -1.0 \end{bmatrix}.$$

You can check the result that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ and $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

15. Euclidean norm. Norm is a generalization of the concept of length and plays an important role in more abstract mathematics (see, e.g., Kreyszig's book on *Functional Analysis* [GenRef7] on p. A1 of App. 1 of the text). We have by (7) on p. 313

$$\left\| [3 \quad 1 \quad -4]^{\mathsf{T}} \right\| = \sqrt{3^2 + 1^2 + (-4)^2} = \sqrt{9 + 1 + 16} = \sqrt{26}.$$

23. Triangle inequality. We have $\mathbf{a} + \mathbf{b} = [3 \quad 1 \quad -4]^{\mathsf{T}} + [-4 \quad 8 \quad -1]^{\mathsf{T}} = [-1 \quad 9 \quad -5]^{\mathsf{T}}$. Thus,

$$\|\mathbf{a} + \mathbf{b}\| = \left\| [-1 \quad 9 \quad -5]^{\mathsf{T}} \right\| = \sqrt{(-1)^2 + 9^2 + (-5)^2} = \sqrt{1 + 81 + 25} = \sqrt{107} = 10.34.$$

Also

$$\|\mathbf{a}\| = \left\| [3 \quad 1 \quad -4]^{\mathsf{T}} \right\| = \sqrt{26} \qquad \text{(from Prob. 15).}$$

$$\|\mathbf{b}\| = \left\| [-4 \quad 8 \quad -1]^{\mathsf{T}} \right\| = \sqrt{(-4)^2 + 8^2 + (-1)^2} = \sqrt{16 + 64 + 1} = \sqrt{81} = 9.$$

Furthermore,

$$\|\mathbf{a}\| + \|\mathbf{b}\| = \sqrt{26} + 9 = 5.099 + 9 = 14.099.$$

The triangle inequality, $\|\mathbf{a} + \mathbf{b}\| \leqq \|\mathbf{a}\| + \|\mathbf{b}\|$, holds for our vectors $\mathbf{a}$, $\mathbf{b}$, since $\|\mathbf{a} + \mathbf{b}\| = 10.34 \leqq \|\mathbf{a}\| + \|\mathbf{b}\| = 14.099$.

Solution for Matrix Multiplication Problem (see p. 110 of this Student Solutions Manual and Study Guide)

$$\mathbf{AB} = \begin{bmatrix} 1 \cdot 10 + 3 \cdot 40 & 1 \cdot 11 + 3 \cdot 30 & 1 \cdot 20 + 3 \cdot 80 & 1 \cdot 50 + 3 \cdot 60 \\ 2 \cdot 10 + 7 \cdot 40 & 2 \cdot 11 + 7 \cdot 30 & 2 \cdot 20 + 7 \cdot 80 & 2 \cdot 50 + 7 \cdot 60 \\ 4 \cdot 10 + 6 \cdot 40 & 4 \cdot 11 + 6 \cdot 30 & 4 \cdot 20 + 6 \cdot 80 & 4 \cdot 50 + 6 \cdot 60 \end{bmatrix}$$

$$= \begin{bmatrix} 10 + 120 & 11 + 90 & 20 + 240 & 50 + 180 \\ 20 + 280 & 22 + 210 & 40 + 560 & 100 + 420 \\ 40 + 240 & 44 + 180 & 80 + 480 & 200 + 360 \end{bmatrix}$$

$$= \begin{bmatrix} 130 & 101 & 260 & 230 \\ 300 & 232 & 600 & 520 \\ 280 & 222 & 560 & 560 \end{bmatrix}.$$

Chap. 8 Linear Algebra: Matrix Eigenvalue Problems

Matrix eigenvalue problems focus on the vector equation

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \tag{1}$$

as described in detail on pp. 322–323. Such problems are so important in applications (Sec. 8.2) and theory (Sec. 8.4) that we devote a whole chapter to them. Eigenvalue problems also appear in Chaps. 4, 11, 12, and 20 (see table on top of p. 323). The amount of theory in this chapter is fairly limited. A modest understanding of linear algebra, which includes matrix multiplication (Sec. 7.2) and row reduction of matrices (Sec. 7.3), is required. Furthermore, you should be able to factor quadratic polynomials.

Sec. 8.1 The Matrix Eigenvalue Problem. Determining Eigenvalues and Eigenvectors

It is likely that you will encounter eigenvalue problems in your career. Thus, make sure that you understand Sec. 8.1 before you move on. Take a look at p. 323. In (1), $\mathbf{A}$ is **any square** (that is, $n \times n$) matrix and it is *given*. Our task is to *find* particular scalar λ's (read "lambda"—don't be intimidated by this Greek letter, it is standard mathematical notation) as well as **nonzero** column vectors $\mathbf{x}$, which satisfy (1). These special scalars and vectors are called *eigenvalues* and *eigenvectors*, respectively.

Example 1 on pp. 324–325 gives a practical example of (1). It shows that the first step in solving a matrix eigenvalue problem (1) by algebra is to set up and solve the "characteristic equation" (4) of $\mathbf{A}$. This will be a quadratic equation in λ if $\mathbf{A}$ is a 2×2 matrix, or a cubic equation if $\mathbf{A}$ is a 3×3 matrix (see **Example 2**, p. 327), and so on. Then, we have to determine the roots of the characteristic equation. These roots are the actual eigenvalues. Once we have found an eigenvalue, we find a corresponding eigenvector by solving a system of linear equations, using Gauss elimination of Sec. 7.3.

Remark on root finding in a characteristic polynomial (4). The main difficulty is finding the roots of (4). Root finding algorithms in numerics (say Newton's method, Sec. 19.2) on your CAS or scientific calculator can be used. However, in a *closed-book* exam where such technology might not be permitted, the following ideas may be useful. For any quadratic polynomials $a\lambda^2 + b\lambda + c = 0$ we have, from algebra, the well-known formula for finding its roots:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{F0}$$

(Note that for the characteristic polynomials $a = 1$, but we did not simplify (F0) for familiarity.) For higher degree polynomials (cubic—degree 3, etc.), the factoring may be possible by some "educated" guessing, aided by the following observations:

F1. The product of the eigenvalues of a characteristic polynomial is equal to the constant term of that polynomial.

F2. The sum of the eigenvalues are equal to $(-1)^{n-1}$ *times* the coefficient of the second highest term of the characteristic polynomial.

Observations F1 and F2 are a consequence of multiplying out factored characteristic equations.

Example of root finding in a characteristic polynomial of degree 3 by "educated guessing." Find the roots ("eigenvalues") of the following cubic characteristic polynomial by solving

$$f(\lambda) = \lambda^3 - 14\lambda^2 + 19\lambda + 210 = 0.$$

F1 suggests that we factor the constant term 210. We get $210 = 1 \cdot 2 \cdot 3 \cdot 5 \cdot 7$. Calculate, starting with the smallest factors (both positive as given) and negative: $f(1) = 216, f(-1) = 176, f(2) = 200,$

$f(-2) = 108, f(3) = 168, f(-3) = 0$. We found an eigenvalue! Thus, $\lambda_1 = -3$ and a factor is $(\lambda + 3)$. We can use long division and apply (F0) (as shown in detail on p. 39 of this Student Solutions Manual). Or we can continue: $f(5) = 80$, $f(-5) = 360$, $f(7) = 0$. Hence $\lambda_2 = 7$. (In case of nonzero f, we would have plugged in -7: then the composite factors $2 \cdot 3$, $-2 \cdot 3$, etc.). Next we apply F2. From F2 we know that the sum of the 3 eigenvalues (roots) must equal $(-1)^2 \cdot (-14) = 14$. Hence $\lambda_1 + \lambda_2 + \lambda_3 = (-3) + 7 + \lambda_3 = 14$, so that $\lambda_3 = 10$. Together, $f(\lambda) = (\lambda + 3)(\lambda - 7)(\lambda + 10)$. *Note that the success of this approach is not guaranteed*, e.g., for eigenvalues being a fraction $\frac{1}{2}$, a decimal 0.194, or a complex number $0.83i$, but it may be useful for average exam questions.

Problem Set 8.1. Page 329

1. Eigenvalues and eigenvectors. Diagonal matrix. For a diagonal matrix the eigenvalues are the main diagonal entries. Indeed, for a general 2×2 diagonal matrix $\mathbf{A}$ we have

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & 0 \\ 0 & a_{22} - \lambda \end{bmatrix}.$$

Then its corresponding characteristic equation is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & 0 \\ 0 & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) = 0.$$

Therefore its roots, and hence eigenvalues, are

$$\lambda_1 = a_{11} \qquad \text{and} \qquad \lambda_2 = a_{22}.$$

Applying this immediately to the matrix given in our problem

$$\mathbf{A} = \begin{bmatrix} 3.0 & 0 \\ 0 & -0.6 \end{bmatrix} \qquad \text{has eigenvalues} \qquad \lambda_1 = 3.0,\ \lambda_2 = -0.6.$$

To determine the eigenvectors corresponding to $\lambda_1 = a_{11}$ of the general diagonal matrix, we have

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{x} = \begin{bmatrix} a_{11} - \lambda_1 & 0 \\ 0 & a_{22} - \lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (a_{11} - \lambda_1)x_1 + 0x_2 \\ 0x_1 + (a_{22} - \lambda_1)x_2 \end{bmatrix}$$
$$= \begin{bmatrix} (a_{11} - \lambda_1)x_1 \\ (a_{22} - \lambda_1)x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Written out in components this is

$$\begin{aligned} (a_{11} - \lambda_1)x_1 &= 0, \\ (a_{22} - \lambda_1)x_2 &= 0. \end{aligned}$$

For our given matrix for $\lambda_1 = 3.0$ we have

$$\begin{aligned} (3.0 - 3.0)x_1 &= 0, &\quad \text{which becomes} \quad 0x_1 &= 0, \\ (-0.6 - 3.0)x_2 &= 0, &\quad \text{which becomes} \quad -3.6x_2 &= 0. \end{aligned}$$

The first equation gives no condition. The second equation gives $x_2 = 0$. Together, for $\lambda_1 = 3.0$, the eigenvectors are of the form $[x_1 \quad 0]^\mathsf{T}$. Since an eigenvector is determined only up to a nonzero constant, we can simply take $[1 \quad 0]^\mathsf{T}$ as an eigenvector. (Other choices could be $[7 \quad 0]^\mathsf{T}$, $[0.37 \quad 0]^\mathsf{T}$, etc. The choices are infinite as you have complete freedom to choose any ***nonzero*** **(!)** value for x_1.)

For the eigenvectors corresponding to $\lambda_2 = a_{22}$ of the general diagonal matrix, we have

$$\begin{aligned}(a_{11} - \lambda_2)x_1 &= 0,\\ (a_{22} - \lambda_2)x_2 &= 0.\end{aligned}$$

For the second eigenvalue $\lambda_2 = -0.6$ of our given matrix, we have

$$\begin{aligned}(3.0 - (-0.6))x_1 &= 0,\\ (-0.6 - (-0.6))x_2 &= 0.\end{aligned}$$

The first equation gives $3.6x_1 = 0$, so that $x_1 = 0$. The second equation $0x_2 = 0$ gives no condition. Hence, by the reasoning above, the eigenvectors corrresponding to $\lambda_2 = -0.6$ are of the form $[0 \quad x_2]^\mathsf{T}$.

We can choose $x_2 = 1$ to obtain an eigenvector $[0 \quad 1]^\mathsf{T}$. We can check our answer by multiplying the matrix by the vector just determined:

$$\mathbf{Ax}_1 = \begin{bmatrix} 3.0 & 0 \\ 0 & -0.6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3.0\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which corresponds to eigenvalue $\lambda_1 = 3.0$, which is correct! Also

$$\mathbf{Ax}_2 = \begin{bmatrix} 3.0 & 0 \\ 0 & -0.6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.6 \end{bmatrix} = (-0.6)\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which corresponds to eigenvalue $\lambda_2 = -0.6$, which is correct!

3. Eigenvalues and eigenvectors. Problem 1 concerned a diagonal matrix, a case in which we could see the eigenvalues immediately. For a general 2×2 matrix the determination of eigenvalues and eigenvectors follows the same pattern. Example 1 on pp. 324–325 illustrates this. For Prob. 3 the matrix is

$$\mathbf{A} = \begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix} \qquad \text{so that} \qquad \mathbf{A} - \lambda I = \begin{bmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{bmatrix}.$$

We calculate the characteristic equation by taking the determinant of $\mathbf{A} - \lambda\mathbf{I}$ and setting it equal to zero. Here

$$\begin{aligned}\begin{vmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{vmatrix} &= (5-\lambda)(-6-\lambda) + 2 \cdot 9\\ &= \lambda^2 + \lambda - 30 + 18\\ &= \lambda^2 + \lambda - 12\\ &= (\lambda + 4)(\lambda - 3) = 0,\end{aligned}$$

and from it the eigenvalues $\lambda_1 = -4$, $\lambda_2 = 3$. Then we find eigenvectors. For $\lambda_1 = -4$ we obtain the system (2), on p. 325 of the textbook [with $a_{11} = 5$, $a_{11} - \lambda_1 = 5 - (-4) = 5 + 4$, $a_{12} = -2$; $a_{21} = 9$, $a_{22} = -6$, $a_{22} - \lambda_1 = -6 - (-4) = -6 + 4$]:

$$\begin{aligned} (5+4)x_1 - \qquad\qquad 2x_2 &= 0 \qquad \text{say,} \qquad x_1 = 2, \quad x_2 = 9 \\ 9x_1 + (-6+4)x_2 &= 0 \qquad \text{(not needed).} \end{aligned}$$

We thus have the eigenvector $\mathbf{x}_1 = [2 \quad 9]^\mathsf{T}$ corresponding to $\lambda_1 = -4$. Similarly, for $\lambda_2 = 3$ obtain the system (2):

$$\begin{aligned} (5-3)x_1 - \qquad\qquad 2x_2 &= 0 \qquad \text{say,} \qquad x_1 = 1, \quad x_2 = 1 \\ 9x_1 + (-6-3)x_2 &= 0 \qquad \text{(not needed).} \end{aligned}$$

We thus have the eigenvector $\mathbf{x}_2 = [1 \quad 1]^\mathsf{T}$ corresponding to $\lambda_2 = 3$. Keep in mind that eigenvectors are determined only up to a nonzero constant factor. Our results can be checked as in Prob. 1.

13. Eigenvalues and eigenvectors. Algebraic multiplicity, geometric multiplicty, and defect of eigenvalues. *Although not needed for finding eigenvalues and eigenvectors, we use this problem to further explain the concepts of algebraic and geometric multiplicity as well as the defect of eigenvalues*. Ordinarily, we would expect that a 3×3 matrix has 3 linearly independent eigenvectors. For symmetric, skew-symmetric, and many other matrices this is true. A simple example is the 3×3 unit matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which has but one eigenvalue, $\lambda = 1$ ($\mathbf{I}$ being a diagonal matrix, so that by Prob. 1 of Sec. 8.1, the eigenvalues can be read off the diagonal). Furthermore, for $\mathbf{I}$, every (nonzero) vector is an eigenvector (since $\mathbf{Ix} = 1 \cdot \mathbf{x}$). Thus, we can choose, for instance, $[1 \quad 0 \quad 0]^\mathsf{T}$, $[0 \quad 1 \quad 0]^\mathsf{T}$, $[0 \quad 0 \quad 1]^\mathsf{T}$ as three representative independent eigenvectors corresponding to $\lambda = 1$.

Let us contrast this to the matrix given in this problem. It has the characteristic equation

$$\begin{aligned} \begin{vmatrix} 13-\lambda & 5 & 2 \\ 2 & 7-\lambda & -8 \\ 5 & 4 & 7-\lambda \end{vmatrix} &= (13-\lambda)\begin{vmatrix} 7-\lambda & -8 \\ 4 & 7-\lambda \end{vmatrix} \\ &\quad - 5\begin{vmatrix} 2 & -8 \\ 5 & 7-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 7-\lambda \\ 5 & 4 \end{vmatrix} \\ &= -\lambda^3 + 27\lambda^2 - 243\lambda + 729 \\ &= -(\lambda - 9)^3 = 0. \end{aligned}$$

We conclude that $\lambda = 9$ is an eigenvalue of algebraic multiplicity 3. We find eigenvectors. We set $\lambda - 9$ in the characteristic matrix to get

$$\mathbf{A} - 9\mathbf{I} = \begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix}.$$

Following the approach of Example 2, p. 327, we row-reduce the matrix just obtained by Gauss elimination (Sec. 7.3) by the following operations:

$$\begin{bmatrix} 4 & 5 & 2 \\ 0 & -\frac{9}{2} & -9 \\ 0 & -\frac{9}{4} & -\frac{18}{4} \end{bmatrix} \begin{array}{l} \text{Pivot Row 1} \\ \text{Row 2 } - \frac{1}{2} \text{ Row 1} \\ \text{Row 3} - \frac{5}{4} \text{ Row 1.} \end{array}$$

Next we pivot Row 2 and perform Row 3 $-\frac{1}{2}$ Row 2 and obtain the following matrix:

$$\begin{bmatrix} 4 & 5 & 2 \\ 0 & -\frac{9}{2} & -9 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that the reduced matrix has rank 2, so we can choose one unknown (one component of an eigenvector) and then determine the other two components. If we choose $x_3 = 1$, then we find from the second row that

$$-\tfrac{9}{2}x_2 - 9x_3 = 0, \qquad x_2 = -\tfrac{2}{9} \cdot 9x_3 = -2x_3 = -2 \cdot 1 = -2.$$

Finally, in a similar vein, we determine from the first row (with substituting $x_2 = -2$ and $x_3 = 1$) that

$$x_1 = -\tfrac{1}{4}(5x_2 + 2x_3) = x_1 = -\tfrac{1}{4}(5 \cdot (-2) + 2 \cdot 1) = -\tfrac{1}{4}(-8) = 2.$$

Together, this gives us an eigenvector $[2 \ \ -2 \ \ \ 1]^{\mathsf{T}}$ corresponding to $\lambda = 9$. We also have that the defect Δ_λ of λ is the algebraic multiplicity M_λ *minus* the geometric multiplicty m_λ (see pp. 327–328 of textbook). As noted above, the algebraic multiplicity for $\lambda = 9$ is 3, since the characteristic polynomial is $-(\lambda - 9)^3$, so that λ appears in a power of 3. The geometric multiplicty for $\lambda = 9$ can be determined as follows. We solved system $\mathbf{A} - 9\mathbf{I} = \mathbf{0}$ by Gauss elimination with back substitution. It gave us only *one* linearly independent eigenvector. Thus, we have that geometric multiplicity for $\lambda = 9$ is 1. [Compare this to Example 2 on p. 327, which gives two independent eigenvectors and hence for a geometric multiplicty of 2 for the eigenvalue -3 (which appeared as a double root and was hence of algebraic multiplicity 2).] Hence for our Prob. 13 we have that the defect Δ_λ for $\lambda = 9$ is

$$\Delta_\lambda = M_\lambda - m_\lambda \qquad \text{gives us} \qquad \Delta_9 = M_9 - m_9 = 3 - 1 = 2.$$

Thus, the defect for $\lambda = 9$ is 2. (By the same reasoning, the defect for the eigenvalue -3 in Example 2, p. 327 of the textbook, is $\Delta_{-3} = M_{-3} - m_{-3} = 2 - 2 = 0$. For the 3×3 unit matrix $\mathbf{I}$ discussed at the beginning, the defect of $\lambda = 1$ is $\Delta_1 = M_{-1} - m_{-1} = 3 - 3 = 0$.)

29. Complex eigenvectors. The reasoning is as follows. Since the matrix is real, which by definition means that its entries are all real, the coefficients of the characteristic polynomial are real, and we know from algebra that a polynomial with real coefficients has real roots or roots that are complex conjugate in pairs.

Sec. 8.2 Some Applications of Eigenvalue Problems

Take a look at the four examples to see precisely how diverse applications lead to eigenvalue problems.

Problem Set 8.2. Page 333

1. Elastic Membrane. We follow the method of Example 1 on p. 330. From the given matrix

$$\mathbf{A} = \begin{bmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{bmatrix}$$

we determine the characteristic equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3.0 - \lambda & 1.5 \\ 1.5 & 3.0 - \lambda \end{vmatrix}$$
$$= (3.0 - \lambda)^2 - 1.5^2 = 1.0 \cdot \lambda^2 - 6.0 \cdot \lambda + 6.75 = 0.$$

Then by (F0) from Sec. 8.1 above

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6.0 \pm \sqrt{(6.0)^2 - 4 \cdot (1.0) \cdot (6.75)}}{2.0}$$
$$= \frac{6.0 \pm \sqrt{9.0}}{2.0} = 3.0 \pm 1.5,$$

so that $\lambda_1 = 3.0 - 1.5 = 1.5$ and $\lambda_2 = 3.0 + 1.5 = 4.5$. First, we deal with eigenvalue $\lambda_1 = 1.5$. A corresponding eigenvector is obtained from one of the two equations

$$(3.0 - 1.5)x_1 + 1.5x_2 = 0, \qquad 1.5x_1 + (3.0 - 1.5)x_2 = 0.$$

They both give us $x_1 = -x_2$, so that we can take $[1 \quad -1]^\mathsf{T}$ as an eigenvector for $\lambda_1 = 1.5$. Geometrically, this vector extends from the origin $(0, 0)$ to the point $(1, -1)$ in the fourth quadrant, thereby making a 315 degree angle in the counterclockwise direction with the positive x-axis. Physically, this means that the membrane from our application is stretched in this direction by a factor of $\lambda_1 = 1.5$.

Similarly, for the eigenvalue $\lambda_2 = 4.5$, we get two equations:

$$(3.0 - 4.5)x_1 + 1.5x_2 = 0, \qquad 1.5x_1 + (3.0 - 4.5)x_2 = 0.$$

Using the first equation, we see that $1.5x_1 = 1.5x_2$, meaning that $x_1 = x_2$ so that we can take $[1 \quad 1]^\mathsf{T}$ as our second desired eigenvector. Geometrically, this vector extends from $(0, 0)$ to $(1, 1)$ in the first quardrant, making a 45 degree angle in the counterclockwise direction with the positive x-axis. The membrane is stretched physically in this direction by a factor of $\lambda_2 = 4.5$.

The figure on the facing page shows a circle of radius 1 and its image under stretching, which is an ellipse. A formula for the latter can be obtained by first stretching, leading from the

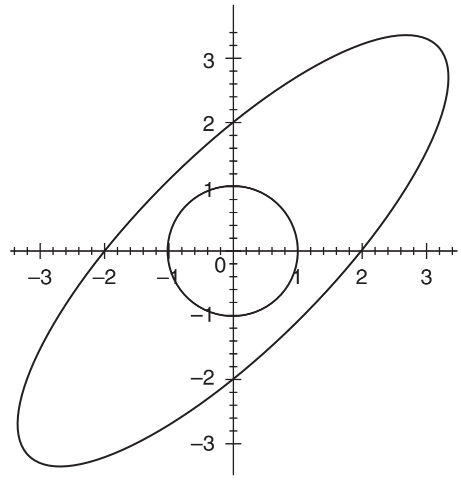
$$\text{circle:} \qquad x_1^2 + x_2^2 = 1,$$

to an

$$\text{ellipse:} \qquad \frac{x_1^2}{4.5^2} + \frac{x_2^2}{1.5^2} = 1,$$

whose axes coincide with the x_1- and the x_2-axes, and then applying a 45 degree rotation, that is, rotation through an angle of $\alpha = \pi/4$, given by

$$u = x_1 \cos\alpha - x_2 \sin\alpha = \frac{x_1 - x_2}{\sqrt{2}}, \qquad v = x_1 \cos\alpha + x_2 \sin\alpha = \frac{x_1 + x_2}{\sqrt{2}}.$$



Sec. 8.2 Prob. 1. Circular elastic membrane stretched to an ellipse

15. Open Leontief input–output model. For reasons explained in the statement of the problem, we have to solve $\mathbf{x} - \mathbf{Ax} = \mathbf{y}$ for $\mathbf{x}$, where $\mathbf{A}$ and $\mathbf{y}$ are given. With the given data we thus have to solve

$$\mathbf{x} - \mathbf{Ax} = (\mathbf{I} - \mathbf{A})\mathbf{x} = \begin{bmatrix} 1-0.1 & -0.4 & -0.2 \\ -0.5 & 1 & -0.1 \\ -0.1 & -0.4 & 1-0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{y} = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix}.$$

For this we can apply the Gauss elimination (Sec. 7.3) to the augmented matrix of the system

$$\begin{bmatrix} 0.9 & -0.4 & -0.2 & 0.1 \\ -0.5 & 1.0 & -0.1 & 0.3 \\ -0.1 & -0.4 & 0.6 & 0.1 \end{bmatrix}.$$

If you carry 6 decimals in your calculation, you will obtain the solution (rounded)

$$x_1 = 0.55, \qquad x_2 = 0.64375, \qquad x_3 = 0.6875.$$

Sec. 8.3 Symmetric, Skew-Symmetric, and Orthogonal Matrices

Remember the definition of such matrices (p. 335). Their complex counterparts will appear in Sec. 8.5.
Consider **Example 1, p. 335:** Notice that the defining properties (1) and (2) can be seen immediately. In particular, for a skew-symmetric matrix you have $a_{jj} = -a_{jj}$, hence the main diagonal entries must be 0.

Problem Set 8.3. Page 338

3. A common mistake. The matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$$

is not skew-symmetric because its main diagonal entries are not 0. Furthermore, the matrix is not orthogonal since $\mathbf{A}^{-1}$ is not equal to $\mathbf{A}^\mathsf{T}$. To determine the spectrum of **A**, that is, the set of all eigenvalues of **A**, we first determine the characteristic equation

$$\begin{vmatrix} 2-\lambda & 8 \\ -8 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 8\cdot(-8)$$
$$= \lambda^2 - 4\lambda + 68 = 0.$$

Its roots are

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4\cdot 68}}{2}$$
$$= \frac{4 \pm \sqrt{-256}}{2.0} = \frac{4 \pm i\sqrt{256}}{2} = \frac{4 \pm i16}{2} = 2 \pm 8i.$$

Thus, the eigenvalues are $\lambda_1 = 2 + 8i$ and $\lambda_2 = 2 - 8i$. An eigenvector for $\lambda_1 = 2 + 8i$ is derived from

$$(2-(2+8i))x_1 + 8x_2 = 0, \qquad -8x_1 + (2-(2+8i))x_2 = 0.$$

Both equations give us that $x_2 = ix_1$. (Note that the second equation gives you $x_2 = (-1/i)x_1$ and you note that $-1 = i^2$ so that $x_2 = (i^2/i)x_1 = ix_1$). Choosing $x_1 = 1$ gives the eigenvector $[1 \quad i]^\mathsf{T}$. Similarly, for $\lambda_2 = 2 - 8i$ we have that

$$(2-(2-8i))x_1 + 8x_2 = 0, \qquad -8x_1 + (2-(2-8i))x_2 = 0.$$

Both equations give us $x_2 = -ix_1$. Choosing $x_1 = 1$ we get the eigenvector $[1 \quad -i]^\mathsf{T}$. Together we have that the spectrum of **A** is $\{2+8i,\ 2-8i\}$. Theorem 1, p. 335, requires that (a) the eigenvalues of a symmetric matrix are real and (b) that the eigenvalues of a skew-symmetric matrix are pure imaginary (that is, of the form bi) or zero. Neither (a) nor (b) is satisfied, as we observed from the outset and are being confirmed by the spectrum and Theorem 1. Theorem 5, p. 337, states that the eigenvalues of an orthogonal matrix are real or complex conjugates in pairs and have absolute value 1. Although our eigenvalues are complex conjugates, their absolute values are $|\lambda_{1,2}| = \sqrt{2^2 + (\pm 8)^2} = \sqrt{68} = 2\sqrt{17} = 8.246 \neq 1$. Hence our observation that **A** is not an orthogonal matrix is in agreement with Theorem 5.

17. Inverse of a skew-symmetric matrix. Let **A** be skew-symmetric, that is,

$$\mathbf{A}^\mathsf{T} = -\mathbf{A}. \tag{1}$$

Let **A** be nonsingular. Let **B** be its inverse. Then

$$\mathbf{AB} = \mathbf{I}. \tag{2}$$

Transposition of (2) and the use of the skew symmetry (1) of **A** give

$$\mathbf{I} = \mathbf{I}^\mathsf{T} = (\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T} = \mathbf{B}^\mathsf{T}(-\mathbf{A}) = -\mathbf{B}^\mathsf{T}\mathbf{A}. \tag{3}$$

Now we multiply (3) by **B** from the right and use (2), obtaining

$$\mathbf{B} = -\mathbf{B}^\mathsf{T}\mathbf{AB} = -\mathbf{B}^\mathsf{T}.$$

This proves that $\mathbf{B} = \mathbf{A}^{-1}$ is skew-symmetric.

Sec. 8.4 Eigenbases. Diagonalization. Quadratic Forms

An outline of this section is as follows. For diagonalizing a matrix we need a basis of eigenvectors (“eigenbasis”). Theorems 1 and 2 (see pp. 339 and 340) tell us of the most important practical cases when such an eigenbasis exists. Diagonalization is done by a similarity transformation (as defined on p. 340) with a suitable matrix **X**. This matrix **X** is constructed from eigenvectors as shown in (5) in Theorem 4, p. 341. Diagonalization is applied to quadratic forms (“transformation to principal axes”) in Theorem 5 on p. 344.

Note that **Example 4** on p. 342 of the textbook and more detailed **Prob. 13** (below) show how to diagonalize a matrix.

Problem Set 8.4. Page 345

1. Preservation of spectrum. We want to prove that

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}, \qquad \mathbf{P} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

have the same spectrum. To compute $\mathbf{P}^{-1}$ by (4*) on p. 304, we need to determine $\det \mathbf{P} = (-4) \cdot (-1) - 2 \cdot 3 = -2$ and get

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}.$$

Next we want to compute $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. First we compute $\mathbf{A}\mathbf{P}$ and get

$$\mathbf{A}\mathbf{P} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -25 & 11 \end{bmatrix}$$

so that

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{P}^{-1}(\mathbf{A}\mathbf{P}) = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -25 & 11 \end{bmatrix} = \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix}.$$

To show the equality of the eigenvalues, we calculate

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) - 16 = \lambda^2 - 25 = (\lambda+5)(\lambda-5) = 0$$

and

$$\det(\hat{\mathbf{A}} - \lambda\mathbf{I}) = \begin{vmatrix} -25-\lambda & 12 \\ -50 & 25-\lambda \end{vmatrix} = (-25-\lambda)(25-\lambda) + 600$$
$$= -625 + \lambda^2 + 600 = \lambda^2 - 25 = (\lambda+5)(\lambda-5) = 0.$$

Since the characteristic polynomials of **A** and $\hat{\mathbf{A}}$ are the same, they have the same eigenvalues. Indeed, by factoring the characteristic polynomials, we see that the eigenvalues of **A** and $\hat{\mathbf{A}}$ are $\lambda_1 = -5$ and $\lambda_2 = 5$. Thus, we have shown that both **A** and $\hat{\mathbf{A}}$ have the same spectrum of $\{-5, 5\}$.

We find an eigenvector $\mathbf{y}_1$ of $\hat{\mathbf{A}}$ for $\lambda_1 = -5$ from

$$(-25 - (-5))x_1 + 12x_2 = 0, \qquad x_1 = 0.6x_2, \qquad \text{say,} \qquad x_2 = 5,\ x_1 = 0.6 \cdot 5 = 3.$$

Thus, an eigenvector $\mathbf{y}_1$ of $\hat{\mathbf{A}}$ is $[3 \quad 5]^\mathsf{T}$. Then we calculate

$$\mathbf{x}_1 = \mathbf{P}\mathbf{y}_1 = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

Similarly, we find an eigenvector $\mathbf{y}_2$ of $\hat{\mathbf{A}}$ for $\lambda_2 = 5$ from

$$(-25 - 5)x_1 + 12x_2 = 0, \qquad x_1 = 0.4x_2, \qquad \text{say,} \qquad x_2 = 5, x_1 = 0.4 \cdot 5 = 2.$$

Thus, an eigenvector $\mathbf{y}_2$ of $\hat{\mathbf{A}}$ is $[2 \quad 5]^\mathsf{T}$. This gives us

$$\mathbf{x}_2 = \mathbf{P}\mathbf{y}_2 = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

To show that indeed $\mathbf{x}_1 = [-2 \quad 4]^\mathsf{T}$ is an eigenvector of $\mathbf{A}$, we compute

$$\mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -20 \end{bmatrix} = -5 \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

Similarly, to show that $\mathbf{x}_2 = [2 \quad 1]^\mathsf{T}$ is an eigenvector of $\mathbf{A}$, we compute

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

These last two vector equations confirm that $\mathbf{x}_1$ and $\mathbf{x}_2$ are indeed eigenvectors of $\mathbf{A}$.

13. Diagonalization. We are given a matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 1 \end{bmatrix}.$$

To find the matrix $\mathbf{X}$ that will enable us to diagonalize $\mathbf{A}$, we proceed as follows.

Step 1. Find the eigenvalues of $\mathbf{A}$. We need to find the characteristic determinant of $\mathbf{A}$, that is,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & 0 & 0 \\ 12 & -2-\lambda & 0 \\ 21 & -6 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 4-\lambda & 0 \\ 12 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(4-\lambda)(-2-\lambda) = (4-\lambda)(-2-\lambda)(1-\lambda) = 0.$$

This gives us the eigenvalues $\lambda_1 = 4$, $\lambda_2 = -2$, and $\lambda_3 = 1$.

Step 2. Find the corresponding eigenvectors of $\mathbf{A}$. We have to solve the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for all the different λ's. Starting with $\lambda_1 = 4$, we have $(\mathbf{A} - 4\mathbf{I})\mathbf{x} = \mathbf{0}$ or

$$\begin{bmatrix} 0 & 0 & 0 \\ 12 & -6 & 0 \\ 21 & -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We form an augmented matrix and solve the system by Gauss elimination with back substitution. We have

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 12 & -6 & 0 & 0 \\ 21 & -6 & -3 & 0 \end{bmatrix},$$

which we rearrange more conveniently as

$$\begin{bmatrix} 21 & -6 & -3 & 0 \\ 12 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using Row 1 as the pivot row, we do the following operation:

$$\begin{bmatrix} 21 & -6 & -3 & 0 \\ 0 & -\frac{18}{7} & \frac{12}{7} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{Pivot Row 1} \\ \text{Row 2 } - \frac{1}{21} \cdot 12 \text{ Row 1.} \\ \\ \end{matrix}$$

Using back substitution we have

$$x_3 = \tfrac{18}{7} \cdot \tfrac{7}{12} x_2 = \tfrac{3}{2} x_2.$$

Substituting this into the first equation yields

$$21x_1 - \tfrac{21}{2}x_2 = 0 \qquad \text{so that} \qquad x_1 = \tfrac{1}{21} \cdot \tfrac{21}{2} x_2 = \tfrac{1}{2} x_2.$$

Thus, we have

$$x_1 = \tfrac{1}{2}x_2 \qquad x_2 \text{ is arbitrary} \qquad x_3 = \tfrac{3}{2}x_2.$$

Choosing $x_2 = 2$, gives $x_1 = 1$ and $x_3 = 3$. This gives an eigenvector $[1 \quad 2 \quad 3]^\mathsf{T}$ corresponding to $\lambda_1 = 4$. We repeat the step for $\lambda_2 = -2$ and get the augmented matrix exchanged:

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 21 & -6 & 3 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & 3 & 0 \end{bmatrix} \begin{array}{l} \text{Pivot Row 1} \\ \text{Row 2 } - 2 \text{ Row 1} \\ \text{Row 3} - \frac{21}{6} \text{ Row 1.} \end{array}$$

Interchange Row 2 and Row 3:

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have $6x_1 = 0$ so that $x_1 = 0$. From the second row we have $-6x_2 + 3x_3 = 0$ so that $x_2 = \frac{1}{2}x_3$. From the last row we see that x_3 is arbitrary. Choosing $x_3 = 2$, we get an eigenvector $[0 \quad 1 \quad 2]^\mathsf{T}$ corresponding to $\lambda_2 = -2$. Finally, for $\lambda_3 = 1$, we get the following augmented matrix with rows conveniently rearranged:

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 12 & -3 & 0 & 0 \\ 21 & -6 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & -6 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{Pivot Row 1} \\ \text{Row 2 } - 4 \text{ Row 1} \\ \text{Row 3} - 7 \text{ Row 1.} \end{array}$$

And, finally, if we do Row 3 − 2 Row 2, then the third row becomes all 0. This shows that x_3 is arbitrary. Furthermore, $x_1 = 0$ and $x_2 = 0$. Choosing $x_3 = 1$, we get an eigenvector $[0 \quad 0 \quad 1]^\mathsf{T}$ corresponding to $\lambda_3 = 1$.

Step 3. Construct the matrix **X** *by writing the eigenvectors of* **A** *obtained in Step 2 as column vectors.* The eigenvectors obtained (written as column vectors) are for

$$\lambda_1 = 4 : \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad \lambda_2 = -2 : \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}; \quad \lambda_3 = 1 : \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{so that} \qquad \mathbf{X} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

Step 4. Determine the matrix $\mathbf{X}^{-1}$. We use the Gauss–Jordan elimination method of Sec. 7.8. We start from $[\mathbf{X} \quad \mathbf{I}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right].$$

Since **A** is lower triangular, the Gauss part of the Gauss–Jordan method is not needed and you can begin with the Jordan elimination of 2, 3, 2 below the main diagonal. This will reduce the given matrix to the unit matrix. Using Row 1 as the pivot row, eliminate 2 and 3. Calculate

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \text{Row 2} - 2 \text{ Row 1} \\ \text{Row 3} - 3 \text{ Row 1.} \end{array}$$

Now eliminate 2 (the only off-diagonal entry left), using Row 2 as the pivot row:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} - 2 \text{ Row 2.} \end{array}$$

The right half of this 3×6 matrix is the inverse of the given matrix. Since the latter has $1 \quad 1 \quad 1$ as the main diagonal, we needed no multiplications, as they would usually be necessary. We thus have

$$\mathbf{X}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

Step 5. Complete the diagonalization process, by computing the diagonal matrix $\mathbf{D} = \mathbf{X}^{-1}\mathbf{AX}$. We first compute

$$\mathbf{AX} = \begin{bmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 8 & -2 & 0 \\ 12 & -4 & 1 \end{bmatrix}.$$

Finally, we compute

$$\mathbf{X}^{-1}\mathbf{AX} = \mathbf{X}^{-1}(\mathbf{AX}) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 8 & -2 & 0 \\ 12 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{D}.$$

Note that, as desired, **D** has the same eigenvalues as **A**. They are the diagonal entries in **D**.

Sec. 8.5 Complex Matrices and Forms. *Optional*

Hermetian, *skew-Hermitian*, and *unitary* matrices are defined on p. 347. They are the complex counterparts of the real matrices (symmetric, skew-symmetric, and orthogonal) on p. 335 in Sec. 8.3. If you feel rusty about your knowledge of complex numbers, you may want to brush up on them by consulting Sec. 13.1, pp. 608–612.

More details on Example 2, p. 347. In matrix **A** the diagonal entries are real, hence equal to their conjugates. $a_{21} = 1 + 3i$ is the complex conjugate of $a_{12} = 1 - 3i$, as it should be for a Hermitian matrix.

In matrix **B** we have $\bar{b}_{11} = \overline{3i} = -3i = -b_{11}$, $\bar{b}_{12} = \overline{2 - i} = -(-2 + i) = -b_{21}$, and $\bar{b}_{22} = \bar{i} = -i = -b_{22}$.

The complex conjugate transpose of **C** is

$$\bar{\mathbf{C}}^{\mathsf{T}} = \begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix}.$$

Multiply this by **C** to obtain the unit matrix. This verifies the defining relationship of a unitary matrix.

Problem Set 8.5. Page 351

5. Hermitian? Skew-Hermitian? Unitary matrix? Eigenvalues and eigenvectors. To test whether the given matrix **A** is Hermetian, we have to see whether $\bar{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}$ (see p. 347). We have to know that the complex conjugate $\bar{z}$ of a complex number $z = a + bi$ is defined as $\bar{z} = a - bi$ (see p. 612 in Sec. 13.1). For a matrix **A**, the conjugate matrix $\bar{\mathbf{A}}$ is obtained by conjugating each element of **A**. Now, since

$$\mathbf{A} = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}, \qquad \bar{\mathbf{A}} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}, \qquad \bar{\mathbf{A}}^{\mathsf{T}} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix},$$

where we used that, for $z = i$, the complex conjugate $\bar{z} = -i$. Clearly, we see that $\bar{\mathbf{A}}^{\mathsf{T}} \neq \mathbf{A}$, so that **A** is not Hermetian.

To test whether **A** is skew-Hermetian, we have to see whether $\bar{\mathbf{A}}^{\mathsf{T}} = -\mathbf{A}$. We notice that

$$\bar{\mathbf{A}}^{\mathsf{T}} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = -\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} = -\mathbf{A}.$$

Thus, **A** is skew-Hermetian.

To test whether **A** is unitary, we have to see whether $\bar{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{-1}$. This can be established by calculating $\bar{\mathbf{A}}^{\mathsf{T}}\mathbf{A}$ and $\mathbf{A}\bar{\mathbf{A}}^{\mathsf{T}}$ and see whether in both cases we get the identity matrix **I**. We have

$$\bar{\mathbf{A}}^{\mathsf{T}}\mathbf{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}.$$

We used that

$$(-i)(i) = -i^2 = -(\sqrt{-1})^2 = -(-1) = 1.$$

Also

$$\mathbf{A}\bar{\mathbf{A}}^{\mathsf{T}} = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}.$$

From this we conclude that, since $\bar{\mathbf{A}}^{\mathsf{T}}\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\bar{\mathbf{A}}^{\mathsf{T}} = \mathbf{I}$, $\bar{\mathbf{A}}^{\mathsf{T}} = \mathbf{A}^{-1}$ and thus **A** is unitary.

To compute eigenvalues, we proceed as usual by computing the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & -\lambda & i \\ 0 & i & -\lambda \end{vmatrix} = (i-\lambda)\begin{vmatrix} -\lambda & i \\ i & -\lambda \end{vmatrix}$$

$$= (i-\lambda)((-\lambda)^2 - i^2) = (i-\lambda)(\lambda^2+1) = (i-\lambda)(\lambda-i)(\lambda+i).$$

From this we get that $\lambda_1 = i$ and $\lambda_2 = -i$. To determine the eigenvector corresponding to $\lambda_1 = i$, we use

$$-ix_2 + ix_3 = 0, \qquad ix_2 - ix_3 = 0.$$

Both equations imply that $ix_2 = ix_3$ so that $x_2 = x_3$, and furthermore x_1 is arbitrary. We can write $x_1 = t, x_2 = x_3, x_3 = x_3$. We can write this in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ x_3 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, we have two independent eigenvectors $[1 \quad 0 \quad 0]^{\mathsf{T}}$ and $[0 \quad 1 \quad 1]^{\mathsf{T}}$ corresponding to eigenvalue $\lambda_1 = i$. Similarly, for $\lambda_2 = -i$ we have

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{so that} \qquad \begin{aligned} 2ix_1 &= 0 \qquad \text{hence} \qquad x_1 = 0 \\ ix_2 + ix_3 &= 0 \qquad \text{hence} \qquad x_2 = -x_3. \end{aligned}$$

Thus, we can write

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

This shows that $[0 \quad -1 \quad 1]^{\mathsf{T}}$ is an eigenvector corresponding to $\lambda_2 = -i$.

9. Hermitian form. To test whether the matrix is Hermetian or skew-Hermetian, we look at the individual elements and their complex conjugates, that is,

$$\begin{aligned} a_{11} &= 4, & \bar{a}_{11} &= 4 = a_{11}, \\ a_{12} &= 3-2i, & \bar{a}_{12} &= 3+2i = a_{21}, \\ a_{21} &= 3+2i, & \bar{a}_{21} &= 3-2i = a_{12}, \\ a_{22} &= 4, & \bar{a}_{22} &= 4 = a_{22}. \end{aligned}$$

Now since

$$\bar{a}_{kj} = a_{jk} \qquad \text{and the elements} \qquad \bar{a}_{jj} = a_{jj} \qquad \text{are real,}$$

we conclude that $\mathbf{A}$ is a Hermetian matrix.

Next we have to calculate (for *complex conjugate numbers*, see p. 612, of textbook)

$$\overline{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\mathbf{x} = [4i \qquad 2-2i]\begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix}\begin{bmatrix} -4i \\ 2+2i \end{bmatrix}.$$

First, we calculate

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix}\begin{bmatrix} -4i \\ 2+2i \end{bmatrix} = \begin{bmatrix} 4(-4i)+(3-2i)(2+2i) \\ (3+2i)(-4i)+(-4)(2+2i) \end{bmatrix}$$
$$= \begin{bmatrix} -16i+(2i+10) \\ (-12i+8)+(-8-8i) \end{bmatrix} = \begin{bmatrix} -14i+10 \\ -20i \end{bmatrix}.$$

Then we calculate

$$\overline{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \overline{\mathbf{x}}^{\mathsf{T}}(\mathbf{A}\mathbf{x}) = [4i \qquad 2-2i]\begin{bmatrix} -14i+10 \\ -20i \end{bmatrix} = (4i)(-14i+10)+(2-2i)(-20i)$$
$$= (-56i^2+40i)+(-40i+40i^2) = -16i^2 = (-16)(-1) = 16.$$

Chap. 9 Vector Differential Calculus. Grad, Div, Curl

Vector calculus is an important subject with many applications in engineering, physics, and computer science as discussed in the opening of Chap. 9 in the textbook. We start with the basics of vectors in R^3 with three real components (and also cover vectors in R^2). We distinguish between *vectors* which geometrically can be depicted as arrows (length and direction) and *scalars* which are just numbers and therefore have no direction. We discuss five main concepts of vectors, which will also be needed in Chap. 10. They are **inner product** (dot product) in Sec. 9.2, **vector product** (cross product) in 9.3, **gradient** in 9.7, **divergence** in 9.8, and **curl** in 9.9. They are quite easy to understand and, as we shall discuss, were invented for their applicability in engineering, physics, geometry, etc. The concept of vectors generalizes to vector functions and vector fields in Sec. 9.4. Similarly for scalars. This generalization allows vector differential calculus in Sec. 9.4, an attractive extension of differential calculus, by the idea of applying regular calculus separately to each component of the vector function.

Perhaps more challenging is the concept of **parametric representation** of curves. It allows curves to be expressed in terms of a *parameter t* instead of the usual *xy*- or *xyz*-coordinates. *Parametrization is very important and will be used here and throughout Chap. 10.* Finally, parameterization leads to arc length.

You need some knowledge of how to calculate third-order determinants (Secs. 7.6, 7.7) and partial derivatives (Sec. A3.2). You should remember the equations for a circle, an ellipse, and some other basic curves from calculus.

Sec. 9.1 Vectors in 2-Space and 3-Space

Some of the material may be familiar to you from Sec. 7.1, pp. 257–261. However, now *we study vectors in the context of* **vector calculus**, instead of linear algebra. Vectors live in **3-space** (R^3) or in **2-space** (R^2, the plane, see p. 309 of Sec. 7.9). Accordingly, the vectors have either three ($n = 3$) or two components. We study the geometry of vectors and use the familiar *xyz*-Cartesian coordinate system. A **vector a** (other notations: $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v}, \mathbf{w}$, etc.) can be thought of as an arrow with a tip, pointing into a direction in 3-space (or 2-space) and having a magnitude (length), as shown in Figs. 164–166, p. 355. The vector has an initial point, say, $P: (x_1, y_1, z_1)$ and terminal point $Q: (x_2, y_2, z_2)$ (see p. 356 and **Prob. 3** below). The vector points in the direction from P to Q. If P is at the origin, then $\mathbf{a} = [a_1, a_2, a_3] = [x_2, y_2, z_2]$.

Other important topics are length $|\mathbf{a}|$ of a vector (see (2), p. 356 and Example 1), vector addition, and scalar multiplication (see pp. 357–358 and **Prob. 15** below).

You also should know about the **standard basis**, $\mathbf{i} = [1, 0, 0], \mathbf{j} = [0, 1, 0], \mathbf{k} = [0, 0, 1]$ of R^3 on p. 359. These **unit vectors** (vectors of length 1) point in the positive direction of the axes of the Cartesian coordinate system and allow us to express any vector **a** in terms of **i**, **j**, **k**, that is,

$$\mathbf{a} = [a_1, a_2, a_3] = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Finally, *distinguish* between **vectors** (magnitude and direction, examples: velocity, force, displacement) and **scalars** (magnitude only(!), examples: time, temperature, length, distance, voltage, etc.).

Problem Set 9.1. Page 360

3. Components and length of a vector. Unit vector. To obtain the components of vector **v** we calculate the differences of the coordinates of its terminal point $Q: (5.5, 0, 1.2)$ minus its initial point $P: (-3.0, 4.0, -0.5)$, as suggested by (1), p. 356. Hence

$$v_1 = 5.5 - (-3.0) = 8.5, \quad v_2 = 0 - 4.0 = -4.0, \quad v_3 = 1.2 - (-0.5) = 1.7.$$

This gives us the desired vector **v**:

$$\mathbf{v} = \overrightarrow{PQ} = [v_1, v_2, v_3] = [8.5, -4.0, 1.7].$$

Sketch the vector so that you see that it looks like an arrow in the *xyz*-coordinate system in space.
The length of the vector **v** [by (2), p. 356] is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_2^2} = \sqrt{(8.5)^2 + (-4.0)^2 + (1.7)^2} = \sqrt{91.14} = 9.547.$$

Finally, to obtain the unit vector **u** in the direction of **v**, we multiply **v** by $1/|\mathbf{v}|$. This scalar multiplication (see (6), p. 358) produces the desired vector as follows:

$$\begin{aligned}\mathbf{u} &= \left(\frac{1}{|\mathbf{v}|}\right)\mathbf{v} = \frac{1}{\sqrt{91.14}}[8.5, -4.0, 1.7]\\ &= \left[\frac{8.5}{\sqrt{91.14}}, -\frac{4.0}{\sqrt{91.14}}, \frac{1.7}{\sqrt{91.14}}\right] = [0.890, -0.419, 0.178].\end{aligned}$$

A remark about the important ***unit vectors*** **i**, **j**, **k**. We can express **v** in terms of **i**, **j**, **k**, the standard basis of R^3 (see p. 359), as follows:

$$\mathbf{v} = [8.5, -4.0, 1.7] = 8.5[1, 0, 0] - 4.0[0, 1, 0] + 1.7[0, 0, 1] = 8.5\mathbf{i} - 4.0\mathbf{j} + \mathbf{1.7k}.$$

This also shows that if we are given $8.5\mathbf{i} - 4.0\mathbf{j} + \mathbf{1.7k}$, then this represents vector $\mathbf{v} = [8.5, -4.0, 1.7]$.

15. Vector addition and scalar multiplication. We claim that it makes no difference whether we first multiply and then add, or whether we first add the given vectors and then multiply their sum by the scalar 7. Indeed, since we add and subtract vectors component-wise, as defined on p. 357 and 7(b), p. 359, we get

$$\mathbf{c} - \mathbf{b} = [5, -1, 8] - [-4, -6, 0] = [5 - (-4), -1 - 6, 8 - 0] = [9, -7, 8].$$

From this we immediately see that

$$7(\mathbf{c} - \mathbf{b}) = 7[9, -7, 8] = [7 \cdot 9, 7 \cdot (-7), 7 \cdot 8] = [63, -49, 56].$$

We calculate

$$\begin{aligned}7\mathbf{c} - \mathbf{7b} &= \mathbf{7}[5, -1, 8] - \mathbf{7}[-4, 6, 0]\\ &= [35, -7, 56] - [-28, 42, 0] = [63, -49, 56]\end{aligned}$$

and get the same result as before. This shows our opening claim is true and illustrates 6(b), p. 358, 7(b), p. 359, and **Example 2** on that page.

27. Forces. Equilibrium. Foremost among the applications that have suggested the concept of a vector were forces, and to a large extent forming the resultant of forces has motivated vector addition. Thus, each of **Probs. 21–25** amounts to the addition of three vectors. "Equilibrium" means that the resultant of the given forces is the zero vector. Hence in Prob. 27 we must determine **p** such that

$$\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{p} = \mathbf{0}.$$

We obtain

$$\begin{aligned}\mathbf{p} = -(\mathbf{u}+\mathbf{v}+\mathbf{w}) &= -\left[-\tfrac{16}{2}+\tfrac{1}{2}-\tfrac{17}{2}, \quad -1+0+1, \quad 0+\tfrac{4}{3}+\tfrac{11}{3}\right] \\ &= [0, \quad 0, \quad -5],\end{aligned}$$

which corresponds to the answer on p. A22.

Sec. 9.2 Inner Product (Dot Product)

Section 9.2 introduces one new concept and a theorem. The **inner product** or **dot product** $\mathbf{a} \cdot \mathbf{b}$ of two vectors is a *scalar*, that is, a number, and is defined by (1) and (2) on p. 361 of the text. **Figure 178**, p. 362 shows that the inner product can be positive, zero, or negative. The case when the inner product is zero is very important. When this happens with two nonzero vectors, then we call the two vectors of the inner product **orthogonal** or perpendicular.

The *Orthogonality Criterion* (**Theorem 1**, p. 362) means that if (i) for two vectors $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$ their dot product $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a} \perp \mathbf{b}$. And (ii) if $\mathbf{a} \perp \mathbf{b}$ with both $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$, then their dot product $\mathbf{a} \cdot \mathbf{b} = 0$. **Problem 1** illustrates the criterion and formula (1), p. 361.

Furthermore, we can use the inner product to express lengths of vectors and angles between them [formulas (3) and (4), p. 362]. This leads to various applications in mechanics and geometry as shown in Examples 2–6, pp. 364–366. Also note that the inner product will be used throughout Chap. 10 in conjunction with new types of integrals.

Problem Set 9.2. Page 367

1. Inner product (dot product). Commutativity. Orthogonality. From the given vectors $\mathbf{a} = [1, 3, -5]$, $\mathbf{b} = [4, 0, 8]$, and $\mathbf{c} = [-2, 9, 1]$ we calculate three dot products as follows:

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= [1, \quad -3, \quad 5] \cdot [4, \quad 0, \quad 8] \\ &= 1 \cdot 4 + (-3) \cdot 0 + 5 \cdot 8] = 4 + 0 + 40 = 44. \qquad \text{(D1)}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \cdot \mathbf{a} &= [4, \quad 0, \quad 8] \cdot [1, \quad -3, \quad 5] \\ &= 4 \cdot 1 + 0 \cdot (-3) + 8 \cdot 5] = 44. \qquad \text{(D2)}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \cdot \mathbf{c} &= [4, \quad 0, \quad 8] \cdot [-2, \quad 9, \quad 1] \\ &= 4 \cdot (-2) + 0 \cdot 9 + 8 \cdot 1 = (-8) + 0 + 8 = 0. \qquad \text{(D3)}\end{aligned}$$

Dot product (D1) has the same value as dot product (D2), illustrating that the dot product is commutative (or symmetric) as asserted in (5b), p. 363 of the text. A general proof of (5b) follows from (2), p. 361, and the commutativity of the multiplication of numbers, $a_1 b_1 = b_1 a_1$, etc. Be aware that the **dot product is commutative**, whereas the **cross product** in the next section **is not commutative**; see in Sec. 9.3, equation (6), p. 370.

Dot product (D3) is zero for two nonzero vectors. From the *Orthogonality Criterion* (Theorem 1, p. 362), explained at the opening of Sec. 9.2 above, we conclude that $\mathbf{b}$ and $\mathbf{c}$ are orthogonal.

9. Linearity of inner product. Since $15\mathbf{a} = [15, \quad -45, \quad 75]$ we have

$$15\mathbf{a} \cdot \mathbf{b} = [15, \quad -45, \quad 75] \cdot [4, \quad 0, \quad 8] = 15 \cdot 4 + (-45) \cdot 0 + 75 \cdot 8 = 660. \qquad \text{(D4)}$$

Similarly,

$$15\mathbf{a} \cdot \mathbf{c} = [15, \quad -45, \quad 75] \cdot [-2, \quad 9, \quad 1] = -30 - 405 + 75 = -360.$$

Their sum is

$$15\mathbf{a}\cdot\mathbf{b} + 15\mathbf{a}\cdot\mathbf{c} = 660 + (-360) = 300.$$

On the other hand, $\mathbf{b} + \mathbf{c} = [4-2, \quad 0+9, \quad 8+1] = [2, \quad 9, \quad 9]$ so that

(D5)
$$\begin{aligned} 15\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= [15, \quad -45, \quad 75]\cdot[2, \quad 9, \quad 9] \\ &= 30 - 405 + 675 = 300. \end{aligned}$$

Equations (D4) and (D5) give the same result because they illustrate linearity. See (5a) (linearity) and (5b) (commutativity p. 363).

15. Parallelogram equality. The proof proceeds by calculation. By linearity (5a) and symmetry (5b), p. 363, we obtain on the left-hand side of (8), p. 363,

$$\begin{aligned} |\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} \\ &= \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b}. \end{aligned}$$

Similarly, for the second term on the left-hand side of (8) we get $\mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b}$. We add both results. We see that the $\mathbf{a}\cdot\mathbf{b}$ terms cancel since they have opposite sign. Hence we are left with

$$2\mathbf{a}\cdot\mathbf{a} + \mathbf{2b}\cdot\mathbf{b} = 2(\mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}) = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2).$$

This is the right side of (8) and completes the proof.

17. Work. This is a major application of inner products and motivates their use. In this problem, the force $\mathbf{p} = [2, 5, 0]$ acts in the displacement from point A with coordinates $(1, 3, 3)$ to point B with coordinates $(3, 5, 5)$. Both points lie in the xyz-plane. Hence the same is true for the segment AB, which represents a "displacement vector" $\mathbf{d}$, whose components are obtained as differences of corresponding coordinates of B minus A. This method of "endpoint minus initial point" was already illustrated in Prob. 3 of Sec. 9.1, which was solved in this Student Solutions Manual. Thus,

$$\mathbf{d} = [d_1, \ d_2, \ d_3] = [3-1, \quad 5-3, \quad 5-3] = [2, \ 2, \ 2].$$

This gives the work as an inner product (a dot product) as in Example 2, p. 364, in the text,

$$W = \mathbf{p}\cdot\mathbf{d} = [2, \ 5, \ 0]\cdot[2, \ 2, \ 2] = 2\cdot 2 + 5\cdot 2 + 0\cdot 2 = 14.$$

Since the third component of the force $\mathbf{p}$ is 0, the force only acts in the xy-plane, and thus the third component of the displacement vector $\mathbf{d}$ does not contribute to the work done.

23. Angle. Use (4), p. 362.

31. Orthogonality. By Theorem 1, p. 362, we know that two nonzero vectors are perpendicular if and only if their inner product is zero (see Prob. 1 above). This means that

$$[a_1, 4, 3]\cdot[3, -2, 12] = 0.$$

Writing out the inner product in components we obtain the linear equation

$$3a_1 + (4)\cdot(-2) + 3\cdot 12 = 0 \quad \text{which simplifies to} \quad 3a_1 - 8 + 36 = 0.$$

Solving for a_1, we get

$$a_1 = -\tfrac{28}{3}.$$

Hence our final answer is that our desired orthogonal vector is $\left[-\tfrac{28}{3},\ 4,\ 3\right]$.

37. Component in a direction of a vector. Components of a vector, as defined in Sec. 9.1, are the components of the vector in the three directions of the coordinate axes. This is generalized in this section. According to (11), p. 365, the component p of $\mathbf{a} = [3, \quad 4, \quad 0]$ in the direction of $\mathbf{b} = [4, \quad -3, \quad 2]$ is

$$p = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|} = \frac{3 \cdot 4 + (4) \cdot (-3) + 2 \cdot 0}{\sqrt{4^2 + (-3)^2 + 2^2}} = \frac{0}{5} = 0.$$

We observe that $p = 0$ and both vectors $\mathbf{a}$ and $\mathbf{b}$ are nonzero. Can you explain what this means geometrically? Our geometric intuition tells us that the only way this is possible is that the two (nonzero) vectors must be orthogonal. Indeed, our intuition is confirmed by mathematics: The numerator of the expression for p is zero and is defined as $\mathbf{a} \bullet \mathbf{b}$. Hence by the Orthogonality Criterion (Theorem 1, p. 362), the two vectors are orthogonal. Thus the component of vector $\mathbf{a}$ in the direction of vector $\mathbf{b}$ is 0. Figure 181 (middle) on p. 365 illustrates our case in two dimensions.

Sec. 9.3 Vector Product (Cross Product)

The **vector product** $\mathbf{a} \times \mathbf{b}$ produces—you guessed it—a *vector*, call it $\mathbf{v}$, that is perpendicular to both vectors $\mathbf{a}$ and $\mathbf{b}$. In addition, the length of $\mathbf{v}$ is equal to the area of the parallelogram whose sides are $\mathbf{a}$ and $\mathbf{b}$. Take a careful look at Fig. 185, p. 369. The parallelogram is shaded in blue. The construct does not work when $\mathbf{a}$ and $\mathbf{b}$ lie in the same straight line or if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$. Carefully look at p. 368. II and IV are the regular case shown in Fig. 185 and I and II the special case, when the construct does not work ($\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{0}$). Since the operation of "vector product" is denoted by a cross "$\times$," it is also called **cross product.** In applications we prefer "right-handed" system coordinate systems, but you have to distinguish between "right-handed" and "left-handed" as explained on p. 369 and Figs. 186–188.

Example 1, p. 370, and **Prob. 11** show how to compute $\mathbf{a} \times \mathbf{b}$ using a "symbolic" third order determinant as given by formula (2**), p. 370, which, by the method of solving determinants (see Sec. 7.6, p. 292), implies (2*), p. 369. Formula (2**) is an easy way to remember (2), p. 368. Carefully read the paragraph "How to Memorize (2)" starting at the bottom of p. 369 and continuing on the next page. It explains **"symbolic" determinant**.

Also be aware that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (minus sign(!): **not commutative** but **anticommutative**, see **Theorem 1**, (6), p. 370, and **Prob. 11**). Furthermore, the cross product is **not associative** (Theorem 1, (7)).

The **scalar triple product** on p. 375 combines the dot product and cross product. **Examples 3–6** and **Prob. 29** emphasize that the cross product and dot product were invented because of many applications in engineering and physics. Cross products and scalar products appear again in Chap. 10.

Problem Set 9.3. Page 374

7. Rotations can be conveniently modeled by vector products, as shown in Example 5, p. 372 of the text. For a clockwise rotation about the y-axis with $\omega = 20\,\text{sec}^{-1}$ the rotation vector $\mathbf{w}$, which always lies in the axis of rotation (if you choose a point on the axis as the initial point of $\mathbf{w}$), is

$$\mathbf{w} = [0,\ 20,\ 0].$$

We have to find the velocity and speed at the point, call it $P : (8, 6, 0)$. From Fig. 192, p. 372, we see that the position vector of the point P at which we want to find the velocity vector $\mathbf{v}$ is vector

$\mathbf{r} = \overrightarrow{OP} = [8, 6, 0]$. From these data the formula (9), p. 372, provides the solution of the equation and formula (2**), p. 370, expands the cross product. Hence the desired velocity (vector) is

$$\mathbf{v} = \mathbf{w} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 20 & 0 \\ 8 & 6 & 0 \end{vmatrix} = \begin{vmatrix} 20 & 0 \\ 6 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ 8 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 20 \\ 8 & 6 \end{vmatrix} \mathbf{k},$$

which we solve by Theorem 2 (b) on p. 297 in Sec. 7.7 and (1), p. 291 in Sec. 7.6. From Theorem 2 we immediately know that the first two determinants in front of $\mathbf{i}$ and $\mathbf{j}$ are 0. The last determinant gives us

$$\begin{vmatrix} 0 & 20 \\ 8 & 6 \end{vmatrix} = 0 \cdot 6 - 20 \cdot 8 = -160.$$

Thus the desired velocity is $\mathbf{v} = [0, 0, -160]$. The speed is the length of the velocity vector $\mathbf{v}$, that is, $|\mathbf{v}| = \sqrt{(-160)^2} = 160$.

11. Vector product (Cross Product). Anticommutativity. From the given vectors $\mathbf{a} = [2, 1, 0]$ and $\mathbf{b} = [-3, 2, 0]$ we calculate the vector product or cross product by (2**), p. 370, denote it by vector $\mathbf{v}$, and get

$$\begin{aligned} \mathbf{v} = \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ -3 & 2 & 0 \end{vmatrix} &= \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 0 \\ -3 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} \mathbf{k} \\ &= (1 \cdot 0 - 0 \cdot 2)\mathbf{i} - (2 \cdot 0 - 0 \cdot (-3))\mathbf{j} + (2 \cdot 2 - 1 \cdot (-3))\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} + 7\mathbf{k} = [0, \quad 0, \quad 7]. \end{aligned}$$

Similarly, if we denote the second desired vector product by $\mathbf{w}$, then

$$\begin{aligned} \mathbf{w} = \mathbf{c} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} &= \begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 0 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 2 \\ 1 & 2 \end{vmatrix} \mathbf{k} \\ &= (2 \cdot 0 - 0 \cdot 2)\mathbf{i} - ((-3) \cdot 0 - 0 \cdot 2)\mathbf{j} + ((-3) \cdot 2 - 2 \cdot 1)\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} - 7\mathbf{k} = [0, \quad 0, \quad -7]. \end{aligned}$$

Finally, the inner product or dot product in components as given by (2) on p. 361 gives us

$$[2, 1, 0] \bullet [-3, 2, 0] = 2 \cdot (-3) + 1 \cdot 2 + 0 \cdot 0 = -6 + 2 + 0 = -4.$$

Comments. We could have computed $\mathbf{v}$ by (2*) on p. 369 instead of (2**). The advantage of (2**) is that it is easier to remember. In that same computation, we could have used Theorem 2(c) on p. 297 of Sec. 7.7 to immediately conclude the first second-order determinant, having a row of zeros, has a value of zero. Similarly for the second-order determinant. For the second cross product, we could have used (6) in Theorem 1(c) on p. 370 and gotten quickly that $\mathbf{w} = \mathbf{c} \times \mathbf{b} = -\mathbf{b} \times \mathbf{c} = -\mathbf{v} = -[0, \quad 0, \quad 7] = [0, \quad 0, \quad -7]$. Since $\mathbf{c} \times \mathbf{b} = -\mathbf{b} \times \mathbf{c}$, the cross product is not commutative but *anticommutative*. This is much better than with matrix multiplication which, in general, was neither commutative nor anticommutative. We could have used these comments to simplify our calculations, but we just wanted to show you that the straightforward approach works.

19. Scalar triple product. The scalar triple product is the most useful of the scalar products that have three or more factors. The reason is its geometric interpretation, shown in Figs. 193 and 194 on p. 374. Using (10), p. 373, and developing the determinant by the third column gives

$$(\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}) = \mathbf{i} \cdot (\mathbf{j} \times \mathbf{k}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot (1 \cdot 1 - 0 \cdot 0) = 1.$$

Note that since the first column of the determinant contained two zeros, the other two determinants did not have to be considered. (If you are familiar with Example 3, on p. 295 in Sec. 9.7, you could have solved the determinant immediately, by noting that it is triangular (even more, diagonal), and thus its value is the product of its diagonal entries). We also are required to calculate

$$(\mathbf{i} \quad \mathbf{k} \quad \mathbf{j}) = \mathbf{i} \cdot (\mathbf{k} \times \mathbf{j}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 \cdot 0 - 1 \cdot 1) = -1.$$

Instead of developing the determinant by the first row, we could have gotten the last result in two more elegant ways. From (16) in Team Project 24 on p. 375 of Sec. 9.3, we learn that $(\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = -(\mathbf{a} \quad \mathbf{c} \quad \mathbf{b})$, which we could have used to evaluate our second scalar triple product. Another possibility would have been to use Theorem 1(a) on p. 295 in Sec. 7.7, which states that the interchange of two rows of a determinant multiplies the value of the determinant by -1.

25. Application of vector product: Moment of a force. This a typical application in mechanics. We are given a force $\mathbf{p} = [2, \quad 3, \quad 0]$ about a point $Q : (2, \quad 1, \quad 0)$ acting on a line through $A : (0, \quad 3, \quad 0)$. We want to find the moment vector $\mathbf{m}$ and its magnitude m. We follow the notation in Fig. 190 on p. 371. Then

$$\mathbf{r} = \overrightarrow{QA} = [0 - 2, \quad 3 - 1, \quad 0 - 0] = [-2, \quad 2, \quad 0].$$

Since we are given the force $\mathbf{p}$, we can calculate the moment vector. Since $\mathbf{r}$ and $\mathbf{p}$ lie in the xy-plane (more precisely: are parallel to this plane, that is, have no z-component), we can calculate the moment vector with $m_1 = 0$, $m_2 = 0$, and

$$m_3 = \begin{vmatrix} -2 & 2 \\ 2 & 3 \end{vmatrix} = (-2) \cdot 3 - 2 \cdot 2 = -10.$$

This is a part of (2**), p. 370, which here looks as follows.

$$\mathbf{m} = \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix}.$$

This "symbolic" determinant is equal to

$$\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 0 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 2 \\ 2 & 3 \end{vmatrix} \mathbf{k}.$$

Since the determinants next to **i** and **j** both contain a column of zeros, these determinants have a value of zero by Theorem 2(b) on p. 297 in Sec. 7.7. The third determinant evaluates to -10 from before. Hence, together, we have

$$\mathbf{m} = -10\mathbf{k} = [0, \quad 0, \quad -10] = [m_1, \quad m_2, \quad m_3],$$

which is in agreement with our reasoning toward the beginning of our problem. It only remains to compute m. We have

$$m = |\mathbf{m}| = |\mathbf{r} \times \mathbf{p}| = \sqrt{0^2 + 0^2 + (-10)^2} = \sqrt{100} = 10.$$

Note that the orientation in this problem is clockwise.

29. Application of vector product: Area of a triangle. The three given points $A : (0, 0, 1)$, $B : (2, 0, 5)$, and $C : (2, 3, 4)$ form a triangle. Sketch the triangle and see whether you can figure out the area directly. We derive two vectors that form two sides of the triangle. We have three possibilities. For instance, we derive **b** and **c** with common initial point A and terminal points B and C, respectively. Then by (1), p. 356, or Prob. 3 of Sec. 9.3 we have

$$\mathbf{b} = \overrightarrow{AB} = [2-0, \quad 0-0, \quad 5-1] = [2, \quad 0, \quad 4],$$
$$\mathbf{c} = \overrightarrow{AC} = [2-0, \quad 3-0, \quad 4-1] = [2, \quad 3, \quad 3].$$

Then [by (2**), p. 370] their vector product is

$$\mathbf{v} = \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 4 \\ 2 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ 3 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 0 \\ 2 & 3 \end{vmatrix} \mathbf{k}$$
$$= (0 \cdot 3 - 4 \cdot 3)\mathbf{i} - (2 \cdot 3 - 4 \cdot 2)\mathbf{j} + (2 \cdot 3 - 0 \cdot 2)\mathbf{k}$$
$$= -12\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} = [-12, \quad 2, \quad 6].$$

Then the length of vector **v** [by (2), p. 357] is

$$|\mathbf{v}| = \sqrt{(-12)^2 + 2^2 + 6^2} = \sqrt{184} = \sqrt{2^3 \cdot 23} = 2 \cdot \sqrt{46}.$$

We also know that vector product $\mathbf{v} = \mathbf{b} \times \mathbf{c}$ is defined in such a way that the length $|\mathbf{v}|$ of vector **v** is equal to the area of the *parallelogram* formed by **b** and **c**, as defined on p. 368 and shown in Fig. 185, p. 369. We also see that the triangle is embedded in the parallelogram in such a way that $\overrightarrow{BC}$ forms a diagonal of the parallelogram and as such cuts the parallelogram precisely into half. Hence the area of the desired triangle is

$$|\mathbf{v}| = \tfrac{1}{2}(2 \cdot \sqrt{46}) = \sqrt{46}.$$

Sec. 9.4 Vector and Scalar Functions and Their Fields. Vector Calculus: Derivatives

The first part of this section (pp. 375–378) explains vector functions and scalar functions. A **vector function** is a function whose values are vectors. It gives a **vector field** in some domain. Take a look at **Fig. 196**, p. 376, **Fig. 198**, p. 378, and the figure of **Prob. 19** to see what vector fields look like. To sketch a vector field by hand, you start with a point P in the domain of the vector function **v**. Then you obtain a vector $\mathbf{v}(P)$ which you draw as vector (an arrow) starting at P and ending at some point, say Q, such that $\overrightarrow{PQ} = \mathbf{v}(P)$. You repeat this process for as many points as you wish. To prevent the vectors drawn from overlapping, you reduce their length proportionally. Thus a typical diagram of vector field gives the correct direction of the vectors but has their length scaled down. Of course, your CAS draws vector fields but sketching a vector field by hand deepens your understanding and recall of the material. A **scalar function** (p. 376) is a function whose values are scalars, that is, numbers. **Example 1**, p. 376, and **Prob. 7** give examples of scalar function.

The second part of this section (pp. 378–380) extends the calculus of a function of one variable to vector functions. Differentiation is done componentwise as given in (10), p. 379. Thus no basically new differentiation rules arise; indeed (11)–(13) follow immediately by writing the products concerned in terms of components (see Prob. 23). Partial derivatives of a vector function are explained on p. 380. Partial derivatives from calculus are reviewed on pp. A69–A71 in Appendix A.

Problem Set 9.4. Page 380

7. Scalar field in the plane. Isotherms. We are given a scalar function $T = T(x, y) = 9x^2 + 4y^2$ that measures the temperature T in a body. *Isotherms* are curves on which the temperature T is constant. (Other curves of the type $f(x, y) = \text{const}$ for a given scalar function f include curves of constant potential ("equipotential lines") and curves of constant pressure ("isobars").) Thus

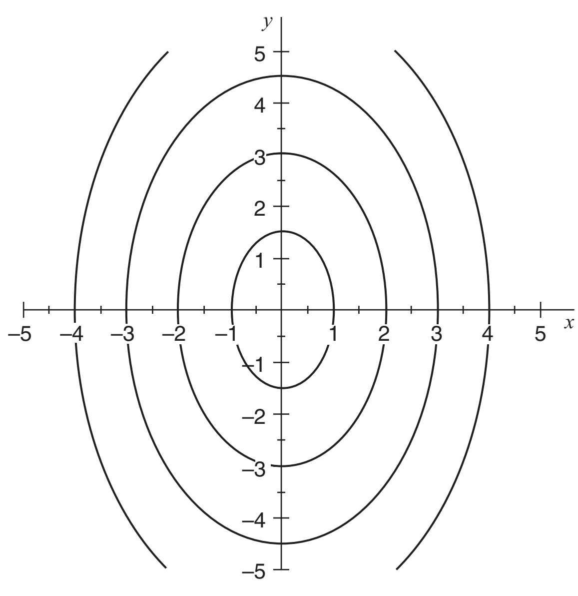
$$9x^2 + 4y^2 = \text{const}.$$

Division by $9 \cdot 4$ gives

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{\text{const}}{36}.$$

Denoting const/36 by a constant c, we can write

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = c.$$



Sec. 9.4 Prob. 7. Some isotherms of the form (E) with $k = \frac{1}{2}, 1, \frac{3}{2}, 2$ of the scalar function T

From analytic geometry, we know that for $c = 1$, this is the standard form of an ellipse. Hence the isotherms are ellipses that lie within each other as depicted above. Note that for any $c \neq 0$, we can divide both sides by c and write it in standard form for ellipses, that is,

$$\frac{x^2}{(2\sqrt{c})^2} + \frac{y^2}{(3\sqrt{c})^2} = 1.$$

This can be further beautified by setting constant $c = k^2$, where $k = \sqrt{c}$ is a constant and obtaining

$$\text{(E)} \qquad \frac{x^2}{(2k)^2} + \frac{y^2}{(3k)^2} = 1.$$

9. Scalar field in space. Level surfaces. We are given a scalar field in space

$$f(x, y, z) = 4x - 3y - 2z$$

and want to know what kind of surfaces the level surfaces are:

$$f(x, y, z) = 4x - 3y - 2z = \text{const.}$$

From Sec. 7.3, Example 1, pp. 273–274, and especially the three-dimensional Fig. 158 we know that for any constant c

$$4x - 3y - 2z = c$$

is a plane. We also know that if we choose different values for c, say $c = 1$, $c = 2$, we obtain a system of two linear equations:

$$\begin{aligned} 4x - 3y - 2z &= 1, \\ 4x - 3y - 2z &= 2. \end{aligned}$$

Such a system of linear equations is inconsistent because it shares no points in common. Thus it represents two parallel planes in 3-space. From this we conclude that in general the level surfaces are parallel planes in 3-space.

19. Vector field. We can express the given vector field as

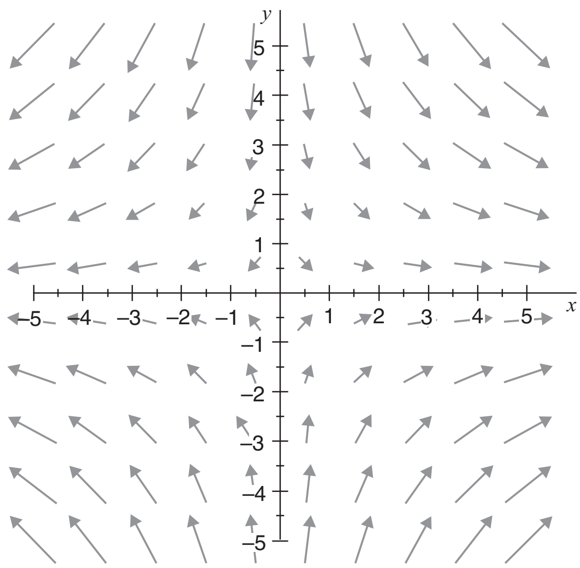
$$\mathbf{v} = \mathbf{v}(x, y, z) = x\mathbf{i} - y\mathbf{j} + 0\mathbf{k}.$$

This shows that the unit vector $\mathbf{k}$ does not make any contribution to $\mathbf{v}$ so that the vectors are parallel to the xy-plane (no z-component). We can write

$$\begin{aligned} \mathbf{v} = x\mathbf{i} - y\mathbf{j} &= x[1, \quad 0, \quad 0] - y[0, \quad 1, \quad 0] \\ &= [x, \quad 0, \quad 0] - [0, \quad y, \quad 0] \\ &= [x, \quad -y, \quad 0]. \end{aligned}$$

Now take any point $P : (x, y, 0)$. Then to graph any such vectors $\mathbf{v}$ would mean that if $\mathbf{v}$ has initial point $P : (x, y, 0)$, $\mathbf{v}$ would go x units in the direction of $\mathbf{i}$ and $-y$ units in the direction of $\mathbf{j}$. Its terminal point would be $Q : (x + x, y - y, 0)$, that is, $Q : (2x, 0, 0)$. Together we have that the vectors are parallel to the xy-plane and, since $y = 0$, have their tip on the x-axis.

Let us check whether our approach is correct. Consider $\overrightarrow{PQ}$. Its components are [by (1), p. 356] $v_1(x, y, z) = 2x - x = x$, $v_2(x, y, z) = 0 - y = -y$, $v_3(x, y, z) = 0 - 0 = 0$, so that by definition of a vector function on p. 376, $\mathbf{v} = \mathbf{v}(x, y, z) = [v_1(x, y, z), \quad v_2(x, y, z), \quad v_3(x, y, z)] =$ $[x, \quad -y, \quad 0]$. But this is precisely the given vector function that defines the vector field!



Sec. 9.4 Prob. 19. Vector field portrayed in two dimensions (as $z = 0$). Note that the direction of the vectors is correct but their length have been reduced, that is, scaled down. This is typical, for otherwise the vectors would overlap.

22–25. Differentiation. Remember that vectors are differentiated componentwise. This applies to derivatives as defined in (10), p. 379, and partial derivatives as defined on p. 380 and shown in Example 5.

23. Vector calculus. Vectors are differentiated by differentiating each of their components separately. To illustrate this, let us show (11), p. 379, that is,

$$(\mathbf{u} \bullet \mathbf{v})' = \mathbf{u}' \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{v}'.$$

Let $\mathbf{u} = [u_1, \quad u_2, \quad u_3]$ and $\mathbf{v} = [v_1, \quad v_2, \quad v_3]$ be two arbitrary vectors in 3-space. Then their dot product [by (2), p. 361] is

$$\mathbf{u} \bullet \mathbf{v} = [u_1, \quad u_2, \quad u_3] \bullet [v_1, \quad v_2, \quad v_3] = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

On the left-hand side (l.h.s.) by first using the sum rule of differentiation and then the product rule we get

$$\begin{aligned}(\mathbf{u} \bullet \mathbf{v})' &= (u_1 v_1 + u_2 v_2 + u_3 v_3)' = (u_1 v_1)' + (u_2 v_2)' + (u_3 v_3)' \\ &= u_1' v_1 + u_1 v_1' + u_2' v_2 + u_2 v_2' + u_3' v_3 + u_3 v_3'.\end{aligned}$$

On the right-hand side (r.h.s.) we have by (10), p. 379, and the product rule

$$\begin{aligned}\mathbf{u}' \bullet \mathbf{v} &= [u_1, \quad u_2, \quad u_3]' \bullet [v_1, \quad v_2, \quad v_3] \\ &= [u_1', \quad u_2', \quad u_3'] \bullet [v_1, \quad v_2, \quad v_3] \\ &= u_1' v_1 + u_2' v_2 + u_3' v_3.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v}' &= [u_1, \quad u_2, \quad u_3] \cdot [v_1, \quad v_2, \quad v_3]' \\ &= [u_1, \quad u_2, \quad u_3] \cdot [v_1', \quad v_2', \quad v_3'] \\ &= u_1 v_1' + u_2 v_2' + u_3 v_3',\end{aligned}$$

so that putting it together and rearranging terms (commutativity of addition of components) gives us the final result:

$$\begin{aligned}\mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' &= u_1' v_1 + u_2' v_2 + u_3' v_3 + u_1 v_1' + u_2 v_2' + u_3 v_3' \\ &= u_1' v_1 + u_1 v_1' + u_2' v_2 + u_2 v_2' + u_3' v_3 + u_3 v_3' = (\mathbf{u} \cdot \mathbf{v})'.\end{aligned}$$

Derive (12) and (13), p. 379, in a similar vein and give two typical examples for each formula. Similarly for the other given vector function.

Sec. 9.5 Curves. Arc Length. Curvature. Torsion

The topic of **parametric representation** on pp. 381–383 *is very important* since in vector calculus many curves are represented in parametric form (1), p. 381, **and parametric representations will be used throughout Chap. 10. Examples 1–4** and **Probs. 5, 13, 17** derive (1) for important curves. Typically, the derivations of such representations make use of such formulas as (5) on p. A64 and others in Sec. A3.1, Appendix 3. For your own studies you may want to start a table of important curves, such as this one:

Curve	Dimension	xy- or xyz-coordinates	Parametric representation	Graph
Circle with center 0 and radius 2	2D	$x^2 + y^2 = 2^2$	$r(t) = [2\cos t, 2\sin t]$	Fig. 201, p. 382
Ellipse with center 0 and axes a, b	2D	$\dfrac{x^2}{a^2} + \dfrac{x^2}{b^2} = 1$	$r(t) = [a\cos t, b\sin t, 0]$	Fig. 202, p. 382
Fill in more curves such as Example 3 (straight line in 3D), Example 4 (circular helix, 3D), etc.				

More curves are in the examples and exercises of other sections in this chapter (and later in Chap. 10). To strengthen your geometric visualization of curves, you may want to sketch the curves. If you get stuck, use your CAS or graphing calculator or even look up the curve by name on the Internet. The table does not have to be complete; it is just for getting used to the material.

Other salient points worth pondering about are:

1. The advantage of parametric representations (1) (p. 381) of curves over other representations. It is absolutely crucial that you understand this completely. Look also at **Figs. 200** and **201**, p. 382, and read pp. 381–383.

2. The distinction between the concepts of **length** (a constant) and **arc length** s (a function). The simplification of formulas resulting from the use of s instead of an arbitrary parameter t. See pp. 385–386.

3. Velocity and acceleration of a motion in space. In particular, look at the basic **Example 7** on pp. 387–388.

4. How does the material in the optional part (pp. 389–390) simplify for curves in the xy-plane? Would you still need the concept of torsion in this case?

Point 4 relates to the beginning of **differential geometry**, a field rich in applications in mechanics, computer-aided engineering design, computer vision and graphics, geodesy, space travel, and relativity theory. More on this area of mathematics is found in Kreyszig's book on *Differential Geometry*, see [GenRef8] on p. A1 of Appendix A.

Problem Set 9.5. Page 390

5. Parametric representation of curve. We want to identify what curve is represented by

$$\mathbf{r}(t) = [x(t), \quad y(t), \quad z(t)] = [2 + 4\cos t, \quad 1 + \sin t, \quad 0].$$

From Example 2, p. 382 we know that

$$\mathbf{q}(t) = [a\cos t, \quad b\sin t, \quad 0]$$

represents an ellipse of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Here we have and extra "2" and "1" in the components $x(t)$ and $y(t)$, respectively. This suggests we try to subtract them as follows:

$$\begin{aligned}\frac{(x(t)-2)^2}{4^2} + \frac{(y(t)-1)^2}{1^2} &= \frac{(2+4\cos t - 2)^2}{4^2} + (1 + \sin t - 1)^2 \\ &= \frac{16\cos^2 t}{4^2} + \sin^2 t = \cos^2 t + \sin^2 t = 1.\end{aligned}$$

It worked and so we see that the parameterized curve represents an ellipse

$$\frac{(x-2)^2}{4^2} + \frac{(y-1)^2}{1^2} = 1,$$

whose center is $(2, 1)$, and whose semimajor axis has length $a = 4$ and lies on the line $x = 2$. Its semiminor axis has length $b = 1$ and lies on the line $y = 1$. Sketch it!

13. Finding a parametric representation. Straight line. The line should go through $A : (2, 1, 3)$ in the direction of $\mathbf{i} + 2\mathbf{j}$. Using Example 3, p. 382, and Fig. 203, p. 383, we see that position vector extending from the origin O to point A, that is,

$$\mathbf{a} = \overrightarrow{OA} = [2, \quad 1, \quad 3]$$

and

$$\mathbf{b} = \mathbf{i} + 2\mathbf{j} = [1, \quad 2, \quad 0].$$

Then a parametric representation for the desired straight line is

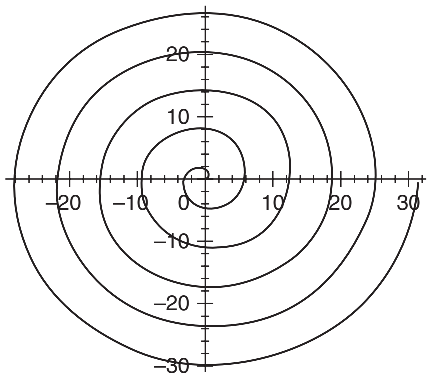
$$\mathbf{r}(t) = \mathbf{a} + t\mathbf{b} = [2, \quad 1, \quad 3] + t[1, \quad 2, \quad 0] = [2 + t, \quad 1 + 2t, \quad 3].$$

17. Parametric representation. Circle. Make a sketch. This curve lies on the elliptic cylinder $\frac{1}{2}x^2 + y^2 = 1$, whose intersection with the xy-plane is the ellipse $\frac{1}{2}x^2 + y^2 = 1, z = 0$. The equation $\frac{1}{2}x^2 + y^2 = 1$ can be written as

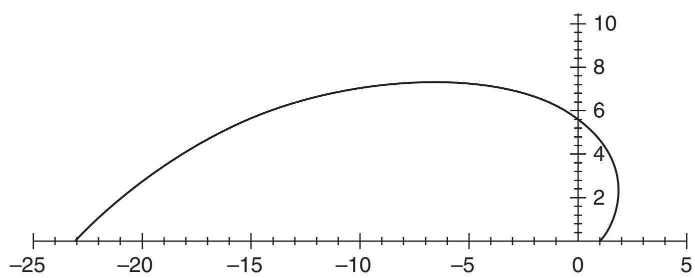
$$\frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{1^2} = 1.$$

This shows that the ellipse has a semimajor axis in the x-direction of length $\sqrt{2}$ and a semiminor axis in the y-direction of length 1. Furthermore, we know that if we intersect (cut) an elliptic cylinder with a plane we get an ellipse. Since the curve lies on the plane $z = y$, its semiaxis in the x-direction has length $\sqrt{2}$ and in the y-direction its semiaxis is 1 (the semiaxis of the cylinder) *times* $\sqrt{2}$. Hence the curve is a circle whose parametrization is given on p. A23. You may want to check the result by substituting $x = \sqrt{2}\cos t$, $y = \sin t$, $z = \sin t$ into the equations of the problem statement.

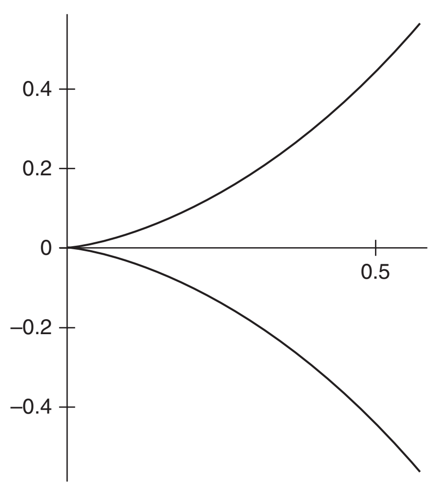
23. CAS PROJECT. Famous curves in polar form. Use your CAS to explore and enjoy these curves. Experiment with these curves by changing their parameters a and b. We numbered the curves



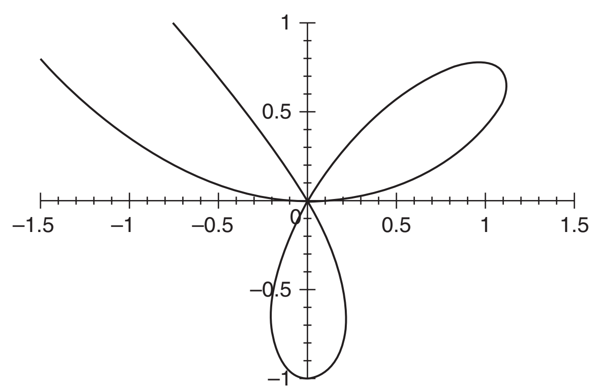
Sec. 9.5 Prob. 23. I. Spiral of Archimedes



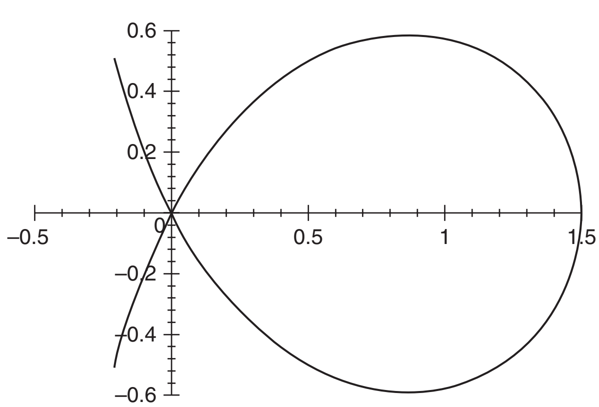
Sec. 9.5 Prob. 23. II. Logarithmic spiral



Sec. 9.5 Prob. 23. III. Cissoid of Diocles



Sec. 9.5 Prob. 23. VI. Folium of Descartes



Sec. 9.5 Prob. 23. VII. Trisectrix of Maclaurin

Prob. 23. **IV**, **V**, and **VII** are not included but are left for the student as an exercise of that CAS PROJECT These graphs are: **IV**. Conchoid of Nicodemus, **V**. Hyperbolic Spiral, and **VIII**. Pascal's snail.

Sec. 9.5 Prob. 23. Remark

I–VIII in the order in which they appear in the problem statement. We selected five representative curves and graphed them. Compare the graphs to your graphs and also provide graphs for the three other curves (**IV**, **V**, **VIII**) not included.

27. Tangent to a curve. We want to find the tangent to

$$\mathbf{r}(t) = [x(t), \quad y(t), \quad z(t)] = [t, \quad 1/t, \quad 0]$$

at point $P: \left(2, \frac{1}{2}, 0\right)$. First, we want to identify the given curve. We note that

$$x(t) \cdot y(t) = 1 \qquad \text{so that} \qquad xy = 1, z = 0.$$

This represents a *hyperbola* in the xy-plane (since $z = 0$). We use the approach of Example 5, pp. 384–385. From (7), p. 384, we know that we get a tangent vector when we take the derivative of $\mathbf{r}(\mathbf{t})$. The tangent vector is

$$\mathbf{r}'(t) = [x'(t), \quad y'(t), \quad z'(t)] = [1, \quad -t^{-2}, \quad 0].$$

The corresponding unit tangent vector determined by (8), p. 384, is

$$\mathbf{u} = \frac{1}{|\mathbf{r}'|}\mathbf{r}' = \left[\frac{1}{\sqrt{1+t^{-4}}}, \quad -\frac{t^{-2}}{\sqrt{1+t^{-4}}}, \quad 0\right],$$

which you may want to verify. Now the point P corresponds to $t = 2$ since

$$\mathbf{r}(2) = [x(2), \quad y(2), \quad z(2)] = \left[2, \quad \tfrac{1}{2}, \quad 0\right].$$

Hence

$$\mathbf{r}'(2) = [1, \quad -2^{-2}, \quad 0] = \left[1, \quad -\tfrac{1}{4}, \quad 0\right].$$

By (9), p. 384, the desired tangent to the curve at P is

$$\begin{aligned}\mathbf{q}(w) = \mathbf{r} + w\mathbf{r}' &= \left[2, \quad \tfrac{1}{2}, \quad 0\right] + w\left[1, \quad -\tfrac{1}{4}, \quad 0\right]\\ &= \left[2+w, \quad \tfrac{1}{2} - \tfrac{1}{4}w, \quad 0\right].\end{aligned}$$

Sketch the hyperbola and the tangent.

29. Length. For the catenary $\mathbf{r} = [t, \quad \cosh t]$ we obtain

$$\mathbf{r}' = [1, \quad \sinh t]$$

and hence

$$\mathbf{r}' \bullet \mathbf{r}' = 1 + \sinh^2 t = \cosh^2 t.$$

Then from (10), p. 385, we calculate the length of the curve from 0 to 1, that is, the given interval $0 \le t \le 1$ to calculate

$$l = \int_a^b \sqrt{\mathbf{r}' \bullet \mathbf{r}'}\,dt = \int_0^1 \cosh t\; dt = \sinh 1 - 0.$$

43. Sun and Earth. *First solution*. See p. A23.
Second solution. This solution is based on the same idea as the first solution. However, it is perhaps slightly more logical and less of a trick. We start from

$$\mathbf{a} = [-R\omega^2 \cos \omega t, \quad -R\omega^2 \sin \omega t] \tag{19}$$

on p. 387. Taking the dot product and then the square root, we have

$$|\mathbf{a}| = R\omega^2.$$

The angular speed is known:

$$\omega = \frac{\text{angle}}{\text{time}} = \frac{2\pi}{T}$$

where $T = 365 \cdot 86{,}400$ is the number of seconds per year. Thus we have

$$|\mathbf{a}| = R\omega^2 = R\left(\frac{2\pi}{T}\right)^2.$$

The length $2\pi R$ of the orbit is traveled in 1 year with a speed of 30 km/sec, as given. This gives us $2\pi R = 30T$, so that

$$R = \frac{30T}{2\pi}. \tag{2}$$

We substitute (2) into (1), cancel R and 2π, and get

$$|\mathbf{a}| = R\omega^2 = \frac{30T}{2\pi}\left(\frac{2\pi}{T}\right)^2 = \frac{30 \cdot 2\pi}{T} = \frac{60\pi}{365 \cdot 86{,}400} = 5.98 \cdot 10^{-6} \text{ [km/sec}^2\text{]}.$$

Comparison. In this derivation we made better use of ω and less of the trick of relating $|\mathbf{a}|$ and $|\mathbf{v}|$ in order to obtain the unknown R. Furthermore, we avoid rounding errors by using numerics only in the last formula where the actual value of T is used. This is in contrast to the first derivation which uses numerics twice. Because of rounding errors (see p. 793 in Chap. 19), the following holds: *Setting up and arranging our formulas and computations in a way that use* ***as little numerics as possible*** *is a good strategy in setting up and solving models.*

Sec. 9.6 Calculus Review: Functions of Several Variables. *Optional*

This section gives the chain rule (**Theorem 1**, p. 393) and the mean value theorem (**Theorem 2**, p. 395) for vector functions of several variables. Its purpose is for reference and for a reasonably self-contained textbook.

Sec. 9.7 Gradient of a Scalar Field. Directional Derivative

This section introduces the important concept of **gradient.** This is the **third** important concept of vector calculus, after **inner product** (or dot product, a scalar, (1), (2), p. 361, in Sec. 9.2), and **vector product** (or cross product, a vector, (2**), p. 370, definition, p. 368, in Sec. 9.3). It is essential that you understand and remember these first two concepts—and now the third concept of gradient—*as you will need them in Chap. 10.* The **gradient of a scalar field** is given by (see Definition 1, p. 396)

$$\mathbf{v} = \operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]. \tag{1}$$

It produces a vector field **v**. Note that ∇f (read **nabla f**) is just a notation. There are three ideas related to the definition of the gradient.

1. **Vector character of the gradient.** A vector **v**, defined in (1) above in terms of components, must have a magnitude and direction independent of components. This is proven in Theorem 1, p. 398.

2. A main application of the gradient grad f is given in (5*), p. 397, in the **directional derivative**, which gives the rate of change of f in any fixed direction. See Definition 2, p. 396, and **Prob. 41**. Special cases are the rates of change $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ in the directions of the coordinate axes.

3. grad f has the direction of the maximum increase of f. It can thus be used as a **surface normal vector** perpendicular to a level surface $f(x, y, z) = \text{const}$. See Fig. 216 and Theorem 2, both on p. 399, as well as **Prob. 33**.

The section ends with potentials of a given vector field on pp. 400–401 and in **Prob. 43**.

Problem Set 9.7. Page 402

13. Electric force. Use of gradients. From $f = \ln(x^2 + y^2)$ we obtain by differentiation (chain rule!)

$$\text{(G)} \qquad \nabla f = \operatorname{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] = \frac{1}{x^2 + y^2}[2x, \quad 2y].$$

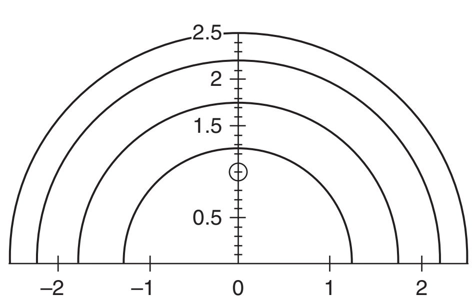
At $P : (8, 6)$ this equals

$$\begin{aligned} \nabla f(P) = \operatorname{grad} f(P) &= \frac{1}{8^2 + 6^2}[2 \cdot 8, \quad 2 \cdot 6] \\ &= [0.16, \quad 0.12]. \end{aligned}$$

Note that this vector has the direction radially outward, from the origin to P because its components are proportional to x, y, z, respectively [see (G)], and have a positive sign. This holds for any point $P \neq (0, 0, 0)$. The length of grad f is increasing with decreasing distance of P from the origin O and approaches infinity as that distance goes to zero.

25. Heat flow. Heat flows from higher to lower temperatures. If the temperature is given by $T(x, y, z)$, the isotherms are the surfaces $T = \text{const}$, and the direction of heat flow is the direction of $-\text{grad}\, T$. For

$$T = \frac{z}{x^2 + y^2}$$



Sec. 9.7 Prob. 25. Isotherms in the horizontal plane $z = 2$

we obtain, using the chain rule,

$$\begin{aligned} -\text{grad } T &= -\left[\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right] \\ &= -\left[-\frac{2xz}{(x^2+y^2)^2}, \quad -\frac{2yz}{(x^2+y^2)^2}, \quad \frac{1}{x^2+y^2}\right] \\ &= \frac{1}{(x^2+y^2)^2}\left[2xz, \quad 2yz, \quad -(x^2+y^2)\right]. \end{aligned}$$

The given point P has coordinates $x = 0, x = 1, z = 2$. Hence at P

$$-\text{grad } T(P) = [0, \quad 4, \quad -1].$$

You may want to sketch the direction of heat flow at P as an arrow. The isotherms are determined by $T = c = \text{const}$ so that their formula is $z = c(x^2 + y^2)$. The figure shows the isotherms in the plane $z = 2$. These are the circles of intersection of the parabola $T = c = \text{const}$ with the horizontal plane $z = 2$. The point P is marked by a small circle on the vertical y-axis.

33. Surface normal. $\nabla f = \text{grad} f$ is perpendicular to the level surfaces $f(x, y, z) = c = \text{const}$, as explained in the text. For the ellipsoid $6x^2 + 2y^2 + z^5 = 225$ we obtain the following variable normal vector:

$$\mathbf{N} = \text{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = [12x, \quad 4y, \quad 2z].$$

At $P : (5, 5, 5)$ the

$$\text{grad} f(P) = [12 \cdot 5, \quad 4 \cdot 5, \quad 2 \cdot 5] = [60, \quad 20, \quad 10],$$

which is one of the surface normal vectors, the other one being $-[60, \quad 20, \quad 10]$.

The length of the normal vector is

$$\begin{aligned} |\mathbf{N}| &= \sqrt{(12x)^2 + (4y)^2 + (2z)^2} = \sqrt{144x^2 + 16y^2 + 4z^2} \\ &= 2\sqrt{36x^2 + 4y^2 + z^2} \end{aligned}$$

and, hence, a unit normal vector by the scalar multiplication by $1/|\mathbf{N}|$,

$$\mathbf{n} = \left(\frac{1}{|\mathbf{N}|}\right)\mathbf{N} = \left(\frac{1}{2\sqrt{36x^2 + 4y^2 + z^2}}\right)[12x, \quad 4y, \quad 2z].$$

Then $-\mathbf{n}$ is the other unit normal vector.

The length of the normal vector at P is

$$2\sqrt{36 \cdot (5^2) + 4 \cdot (5)^2 + 5^2} = 2 \cdot 5\sqrt{36 + 4 + 1} = 10\sqrt{41},$$

so that a unit vector at P is

$$\mathbf{n} = \left(\frac{1}{|\mathbf{N}|}\right)\mathbf{N} = \frac{1}{10\sqrt{41}}[60, \quad 20, \quad 10] = \frac{1}{\sqrt{41}}[6, \quad 2, \quad 1].$$

41. **Directional derivative.** The directional derivative gives the rate of change of a scalar function f in the direction of a vector $\mathbf{a}$. In its definition (5*), p. 397, the gradient ∇f gives the maximum rate of change, and the inner product of ∇f and the unit vector $(1/|\mathbf{a}|)\,\mathbf{a}$ in the desired direction gives the desired rate of change (5*). This one finally evaluates at the given point.

Hence from $f = xyz$, $\mathbf{a} = [1, \quad -1, \quad 2]$, and $P : (-1, 1, 3)$ we calculate

$$|\mathbf{a}| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$$

$$\mathbf{v} = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = [yz, \quad xz, \quad xy],$$

and

$$\frac{1}{|\mathbf{a}|}\mathbf{v} \bullet \mathbf{a} = \frac{1}{|\mathbf{a}|}[yz, \quad -xz, \quad xy] \bullet [1, \quad 1, \quad 2]$$
$$= \frac{1}{\sqrt{6}}(yz - xz + 2xy).$$

Evaluating this at P, we obtain

$$D_a f(P) = \frac{1}{\sqrt{6}}(3 \cdot 2 - (-1)3 + 2(-1) \cdot 1) = \frac{4}{\sqrt{6}} = \sqrt{\frac{8}{3}}.$$

43. **Potential.** Instead of just using inspection, we want to show you a systematic way of finding a potential f. This method will be discussed further in Sec. 10.2. We are taking the liberty of looking a little bit ahead. Using the method of Example 2, p. 421, of Sec. 10.2, we have

$$\mathbf{v}(x, y, z) = [v_1(x), \quad v_2(y), \quad v_3(z)] = [yz, \quad xz, \quad xy].$$
$$\mathbf{v} = \operatorname{grad} f = [f_x, \quad f_y, \quad f_z].$$

We take the partials

(A) $$f_x = yz.$$
(B) $$f_y = xz.$$
(C) $$f_z = xy.$$

Integrating (A) gives

(D) $$f = \int yz\,dx = yz \int dx = yzx + g(y, z).$$

Taking the partial of (D) with respect to y and using (B) gives

(E) $$f_y = zx + g_y(y, z) = xz \qquad \text{so that} \qquad g_y(y, z) = 0.$$

Hence

$$g(y, z) = h(z) + C_1 \quad (C_1 \text{ a constant}).$$

Substituting this into (D) gives

$$f = yzx + h(z) + C_1.$$

Taking the partial with respect to z and using (C) gives

$$f_z = xy + h'(z) = xy \qquad \text{so that} \qquad h'(z) = 0.$$

Hence $h(z) =$ constant. Thus the potential is

$$f = xyz + C \quad (C \text{ a constant}).$$

Compute f_x, f_y, f_z and verify that our answer is correct. Since C is arbitrary, we can choose $C = 0$ and get the answer on p. A24.

Sec. 9.8 Divergence of a Vector Field

Divergence is a scalar function and is the *fourth* concept of note (following gradient, a vector function of Sec. 9.7, and the others). To calculate divergence div $\mathbf{v}$ by (1), p. 403 for a vector function $\mathbf{v} = \mathbf{v}(x, y, z) = [v_1(x, y, z), \quad v_2(x, y, z), \quad v_3(x, y, z)]$ we use partial differentiation as in calculus to get

$$\text{(1)} \qquad \operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

For a *review of* **partial differentiation** consult Appendix 3, Sec. A3.2, pp. A69–A71.

Divergence plays an important role in Chap. 10. Whereas the **gradient** obtains a vector field $\mathbf{v}$ from a scalar field f, the **divergence** operates in the opposite sense, obtaining a scalar field (1) div $\mathbf{v}$ from a vector field v. Of course, grad and div are not inverses of each other; they are entirely different operations, created because of their applicability in physics, geometry, and elsewhere. The physical meaning and practical importance of the divergence of a vector function (a vector field) are explained on pp. 403–405. Applications are in fluid flow (Example 2, p. 404) and other areas in physics.

Problem Set 9.8. Page 406

1. Divergence. The calculation of div $\mathbf{v}$ by (1) requires that we take the partial derivatives of each of the components of $\mathbf{v} = [x^2, \quad 4y^2, \quad 9z^2]$ and add them. We have

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 2x + 8y + 18z.$$

Details: $v_1 = x^2$, $\partial v_1/\partial x = 2x$, etc.

The value at $P: \left(-1, 0, \frac{1}{2}\right)$ is

$$\operatorname{div} \mathbf{v} = 2 \cdot (-1) + 8 \cdot 0 + 18 \cdot \tfrac{1}{2} = 7.$$

9. PROJECT. Formulas for the divergence. These formulas help in simplifying calculations as well as in theoretical work. They follow by straightforward calculations directly from the definitions. For instance, for (b), by the definition of the divergence and by product differentiation you obtain

$$\begin{aligned} \operatorname{div}(f\mathbf{v}) &= (fv_1)_x + (fv_2)_y + (fv_3)_z \\ &= f_x v_1 + f_y v_2 + f_z v_3 + f[(v_1)_x + (v_2)_y + (v_3)_z] \end{aligned}$$

$$= (\text{grad} f) \bullet \mathbf{v} + f \text{ div } \mathbf{v}$$
$$= f \text{ div } \mathbf{v} + \mathbf{v} \bullet \nabla f.$$

11. Incompressible flow. The velocity vector is $\mathbf{v} = y\,\mathbf{i} = [y, 0, 0]$. Hence div $\mathbf{v} = 0$. This shows that the flow is incompressible; see (7) on p. 405. $\mathbf{v}$ is parallel to the x-axis. In the upper half-plane it points to the right and in the lower half-plane it points to the left. On the x-axis ($y = 0$) it is the zero vector. On each horizontal line $y = \text{const}$ it is constant. The speed is larger the farther away from the x-axis we are. From $\mathbf{v} = y\mathbf{i}$ and the definition of a velocity vector you obtain

$$\mathbf{v} = \left[\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt}\right] = [y, \quad 0, \quad 0].$$

This vector equation gives three equations for the corresponding components,

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = 0, \quad \frac{dz}{dt} = 0.$$

Integration of $dz/dt = 0$ gives

$$z(t) = c_3 \quad \text{with } c_3 = 0 \text{ for the face } z = 0, \quad c_3 = 1 \text{ for the face } z = 1$$

and $0 < c_3 < 1$ for particles inside the cube. Similarly, by integration of $dy/dt = 0$ you obtain $y(t) = c_2$ with $c_2 = 0$ for the face $y = 0$, $c_2 = 1$ for the face $y = 1$ and $0 < c_2 < 1$ for particles inside the cube.

Finally, $dx/dt = y$ with $y = c_2$ becomes $dx/dt = c_2$. By integration,

$$x(t) = c_2\,t + c_1.$$

From this,

$$x(0) = c_1 \text{ with } c_1 = 0 \text{ for the face } x = 0, \qquad c_1 = 1 \text{ for the face } x = 1.$$

Also

$$x(1) = c_1 + c_2.$$

Hence

$$x(1) = c_2 + 0 \text{ for the face } x = 0, \qquad x(1) = c_2 + 1 \text{ for the face } x = 1$$

because $c_1 = 0$ for $x = 0$ and $c_1 = 1$ for $x = 1$, as just stated. This shows that the distance of these two parallel faces has remained the same, namely, 1. And since nothing happened in the y- and z-directions, this shows that the volume at time $t = 1$ is still 1, as it should be in the case of incompressibility.

Sec. 9.9 Curl of a Vector Field

The **curl** of a vector function (p. 406) is the *fifth* concept of note. Just as grad and div, the curl is motivated by physics and to a lesser degree by geometry. In Chap. 10, we will see that the curl will also play a role in integration. In the definition of curl we use the concept of a *symbolic determinant*, encountered in Sec. 9.3

(see (2**), p. 370). The curl $\mathbf{v}$ of a vector function $\mathbf{v} = [v_1(x, y, z), \quad v_2(x, y, z), \quad v_3(x, y, z)]$ is defined by (with all gaps filled)

$$
\text{(1)} \qquad \operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}
$$

$$
= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ v_1 & v_2 \end{vmatrix} \mathbf{k}
$$

$$
= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \mathbf{i} - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) \mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \mathbf{k}
$$

$$
= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \mathbf{k}.
$$

Details. In (1), the first equality came from the definition of curl. The second equality used the definitions of cross product and nabla $\nabla = [\partial/\partial x, \partial/\partial y, \partial/\partial z]$ in setting up the symbolic determinant (recall (2**), p. 370). To obtain the fourth equality we developed the symbolic determinant by row 1 and by the checkerboard pattern of signs of the cofactors

$$
+ \qquad - \qquad +
$$

as shown on page 294. *It is here that we would stop and indeed you can use the fourth equality to compute* curl $\mathbf{v}$. However, the mathematical literature goes one step further and uses the fifth equality as the definition of the curl. In this step, we absorb the minus sign in front of $(\ldots)\mathbf{j}$, that is,

$$
-\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) \mathbf{j} = +\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \mathbf{j}.
$$

To remember the formula for the curl, you should know the first two equalities and how to develop a third-order determinant. Note that (1) defines the right-handed curl. (A left-handed curl is defined by putting a minus sign in front of the third-order symbolic determinant.)

Problem Set 9.9. Page 408

5. **Calculation of the curl.** For the calculations of the curl use (1), where the "symbolic determinant" helps one to memorize the actual formulas for the components given in (1) below the determinant. We are given that

$$
v_1 = x^2 yz, \quad v_2 = xy^2 z, \quad v_3 = xyz^2.
$$

With this we obtain from (1) for the components of, say $\mathbf{a} = \operatorname{curl} \mathbf{v}$

$$
\begin{aligned}
a_1 &= (v_3)_y - (v_2)_z = (xyz^2)_y - (xy^2 z)_z = xz^2 - xy^2, \\
a_2 &= (v_2)_z - (v_3)_x = (x^2 yz)_z - (xyz^2)_x = x^2 y - yz^2, \\
a_3 &= (v_2)_x - (v_1)_y = (xy^2 z)_x - (x^2 yz)_y = y^2 z - x^2 z.
\end{aligned}
$$

This corresponds to the answer on p. A24. For practice you may want to fill in the details of using the symbolic determinant to get the answer.

11. Fluid flow. Both div and curl characterize essential properties of flows, which are usually given in terms of the velocity vector field $\mathbf{v}(x, y, z)$. The present problem is two-dimensional, that is, in each plane $z = \text{const}$ the flow is the same. The given velocity is

$$\mathbf{v} = [\,y, \quad -2x, \qquad 0].$$

Hence div $\mathbf{v} = 0 + 0 + 0 = 0$. This shows that the flow is incompressible (see the previous section). Furthermore, from (1) in the present section, we see that the first two components of the curl are zero because they consist of expressions involving v_3, which is zero, or involving the partial derivative with respect to z, which is zero because $\mathbf{v}$ does not depend on z. There remains

$$\operatorname{curl} \mathbf{v} = ((v_2)_x - (v_1)_y)\mathbf{k} = (-2 - 1)\mathbf{k} = -3\mathbf{k}.$$

This shows that the fluid flow is not irrotational. Now we determine the paths of the particles of the fluid. From the definition of velocity we have

$$v_1 = \frac{dx}{dt}, \quad v_2 = \frac{dy}{dt}.$$

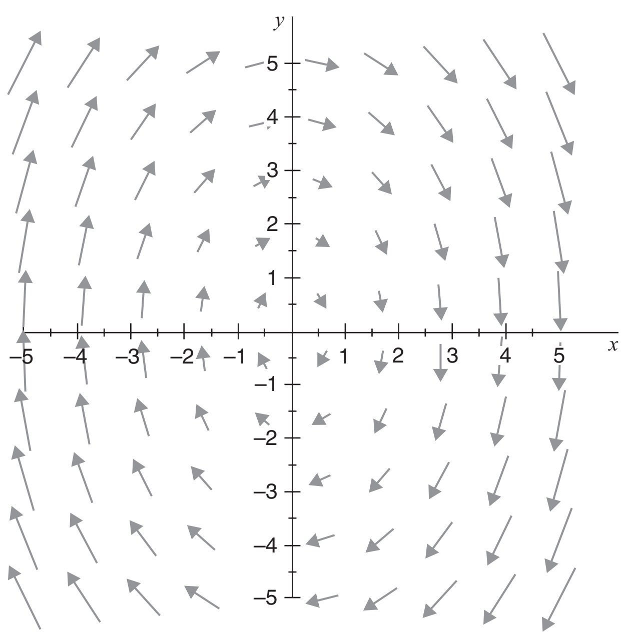
From this and the given velocity vector $\mathbf{v}$ we see that

$$\text{(A)} \qquad \frac{dx}{dt} = y$$

$$\text{(B)} \qquad \frac{dy}{dt} = -2x.$$

This system of differential equations can be solved by a trick worth remembering. The right side of (B) times the left side of (A) is $-2x(dx/dt)$. This must equal the right side of (A) times the left side of (B), which is $y(dy/dt)$. Hence

$$-2x\frac{dx}{dt} = y\frac{dy}{dt}.$$



Sec. 9.9 Prob. 11. Vector field $\mathbf{v} = [\,y, -2x]$ of fluid flow in two dimensions (as $z = 0$).

We can now integrate with respect to t on both sides and multiply by -2, obtaining

$$\int -2x\frac{dx}{dt}dt = \int y\ \frac{dy}{dt}dt$$
$$-2\int x\,dx = \int y\,dy$$
$$-2\frac{x^2}{2} = \frac{y^2}{2} + \widetilde{C}.$$

Here $\widetilde{C}$ is a constant. This can be beautified. If we set $\widetilde{C}$ equal to the square of another constant, say, C^2, then

$$x^2 + \frac{y^2}{2} = C^2 \quad (\text{where } \widetilde{C} = C^2)$$
$$\frac{x^2}{C^2} + \frac{y^2}{(\sqrt{2}C)^2} = 1.$$

This shows that the paths of the particles (the streamlines of the flow) are ellipses with a common center 0:

$$\frac{x^2}{C^2} + \frac{y^2}{(\sqrt{2}C)^2} = 1.$$

Note that the z-axis is the axis of rotation.

Can you see that the vector field is not irrotational? Can you see the streamlines of the fluid flow?

Chap. 10 Vector Integral Calculus. Integral Theorems

We continue our study of vector calculus started in Chap. 9. In this chapter we explore **vector *integral* calculus**. The use of vector functions introduces two new types of integrals, which are the *line integral* (Secs. 10.1, 10.2) and the *surface integral* (Sec. 10.6) and relates these to the more familiar *double integrals* (see Review Sec. 10.3, Sec. 10.4) and *triple integrals* (Sec. 10.7), respectively. Furthermore, Sec. 10.9 relates surface integrals to line integrals. The roots of these integrals are largely physical intuition. The main theme underlying this chapter is the **transformation (conversion) of integrals into one another**. What does this mean? Whenever possible, we want to obtain a new integral that is easier to solve than the given integral.

Vector integral calculus is very important in engineering and physics and has many applications in mechanics (Sec. 10.1), fluid flow (Sec. 10.8), heat problems (Sec. 10.8), and in other areas.

Since this chapter covers a substantial amount of material and moves quite quickly, you may need to allocate more study time for this chapter than the previous one. It will take time and practice to get used to the different integrals.

You should remember from Chap. 9 (see chapter summary on pp. 410–412 of textbook): parametric representations of curves (p. 381 of Sec. 9.5), dot product, gradient, curl, cross product, and divergence. Reasonable knowledge of double integrals (reviewed in Sec. 10.3) and partial derivatives (reviewed on pp. A69–A71 in App. A3.2 of the textbook) is required. It is also helpful if you recall some of the 3D objects from calculus, such as a sphere, a cylinder, etc. It may be useful to continue working on your table of parametric representations (see p. 156 of Student Solutions Manual).

Sec. 10.1 Line Integrals

The first new integral, the **line integral** (3), p. 414, generalizes the definite integral from calculus. Take a careful look at (3). Instead of integrating a function f along the x-axis we now integrate a vector function $\mathbf{F}$ over a curve C from a point A to a point B. The right side of (3) shows us how to convert such a line integral into a definite integral with t as the variable of integration. Furthermore, t is the parameter in C, which is represented in parametric form $\mathbf{r}(t)$ (recall (1), p. 381, in Sec. 9.5). Typically, the first step is to find such a parametric representation. Then one has to form the dot product (see (2), p. 361, of Sec. 9.2) consisting of $\mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)$. **Example 1**, p. 415, and **Prob. 5** show, in detail, how to solve (3) when C is in the *plane* and **Example 2**, p. 415, and **Prob. 7** when C is in *space*. Other versions of the line integral are (8), p. 417 (Example 5 and Prob. 15) and (8*) (Prob. 19). An important application is work done by force, pp. 416–417.

Notation: $\int_C$ (p. 414) for regular line integral; $\oint_C$ (p. 415) for line integral when C is a closed curve (e.g., circle, starting at point A and ending at point B coincide).

The approach of *transforming* (*converting) one integral into another integral* is prevalent throughout this chapter and needs practice to be understood fully.

The advantage of vector methods is that methods in space and methods in the plane are very similar in most cases. Indeed, the treatment of line integrals in the plane carries directly over to line integrals in space, serving as a good example of the *unifying principles of engineering mathematics* (see Theme 3 of p. ix in the Preface to the textbook).

Problem Set 10.1. Page 418

5. Line integral in the plane. The problem parallels Example 1, p. 415, in the text. Indeed, we are setting up our solution as a line integral to be evaluated by (3) on p. 414. The quarter-circle C, with the center the origin and radius 2, can be represented by

(I) $\qquad \mathbf{r}(t) = [2\cos t, \quad 2\sin t], \qquad \text{in components,} \quad x = 2\cos t, \quad y = 2\sin t,$

where t varies from $t = 0$ [the initial point $(2, 0)$ of C on the x-axis] to $t = \pi/2$ [the terminal point $(0, 2)$ of C]. We can show, by substitution, that we get the correct terminal points:

$$\text{for } t = 0 \qquad \text{we have} \qquad \mathbf{r}(t) = \mathbf{r}(0) = [2\cos 0, \quad 2\sin 0] = [2 \cdot 1, \quad 2 \cdot 0] = [2, \quad 0],$$

giving us the point $(2, 0)$. Similarly for the other terminal point. (Note that the path would be a full circle, if t went all the way up to $t = 2\pi$.)

The given function is a vector function

$$\text{(II)} \qquad \mathbf{F} = [xy, \quad x^2y^2].$$

$\mathbf{F}$ defines a vector field in the xy-plane. At each point (x, y) it gives a certain vector, which we could draw as a little arrow. In particular, at each point of C the vector function $\mathbf{F}$ gives us a vector. We can obtain these vectors simply by substituting x and y from (I) into (II). We obtain

$$\text{(III)} \qquad \mathbf{F}(\mathbf{r}(t)) = [4\cos t \sin t, \quad 16\cos^2 t \sin^2 t].$$

This is now a vector function of t defined on the quarter-circle C.

Now comes an important point to observe. We do not integrate $\mathbf{F}$ itself, but we integrate the dot product of $\mathbf{F}$ in (III) and the tangent vector $\mathbf{r}'(t)$ of C. This dot product $\mathbf{F} \bullet \mathbf{r}'$ can be "visualized" because it is the component of $\mathbf{F}$ in the direction of the tangent of C (times the factor $|\mathbf{r}'(t)|$), as we can see from (11), p. 365, in Sec. 9.2, with $\mathbf{F}$ playing the role of $\mathbf{a}$ and $\mathbf{r}'$ playing the role of $\mathbf{b}$. Note that if t is the arc length s of C, then $\mathbf{r}'$ is a unit vector, so that that factor equals 1, and we get exactly that tangential projection. Think this over before you go on calculating.

Differentiation with respect to t gives us the tangent vector

$$\mathbf{r}'(t) = [-2\sin t, \quad 2\cos t].$$

Hence the dot product is

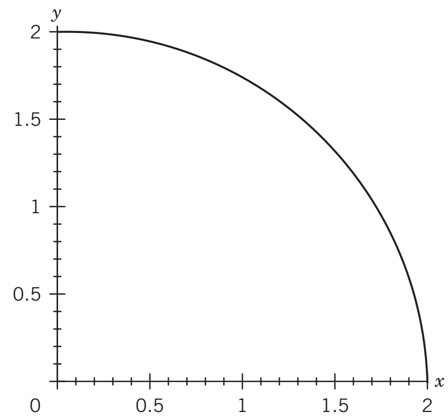
$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) &= [4\cos t \sin t, \quad 16\cos^2 t \sin^2 t] \bullet [-2\sin t, \quad 2\cos t] \\
&= 4\cos t \sin t(-2\sin t) + 16\cos^2 t \sin^2 t(2\cos t) \\
\text{(IV)} \qquad &= -8\cos t \sin^2 t + 32\cos^3 t \sin^2 t \\
&= -8\cos t \sin^2 t + 32\cos t(1 - \sin^2 t)\sin^2 t \\
&= -8\sin^2 t \cos t + 32\sin^2 t \cos t - 32\sin^4 t \cos t.
\end{aligned}$$

Now, by the chain rule,

$$(\sin^3 t)' = 3\sin^2 t \cos t, \quad (\sin^5 t)' = 5\sin^4 t \cos t$$

so that the last line of (IV) can be readily integrated. Integration with respect to t (the parameter in the parametric representation of the path of integration C) from $t = 0$ to $t = \pi/2$ gives

$$\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} &= \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\
&= -8\int_0^{\pi/2} \sin^2 t \cos t\, dt + 32\int_0^{\pi/2} \sin^2 t \cos t\, dt - 32\int_0^{\pi/2} \sin^4 t \cos t\, dt \\
&= \left[-\frac{8}{3}\sin^3 t + \frac{32}{3}\sin^3 t - \frac{32}{5}\sin^5 t\right]_0^{\pi/2}.
\end{aligned}$$



Sec. 10.1 Prob. 5. Path of integration in the xy-plane

At $t = 0$, the sine is 0. At the upper limit of integration $t = \pi/2$, the sine is 1. Hence the result is

$$= -\tfrac{8}{3} + \tfrac{32}{3} - \tfrac{32}{5} = \tfrac{8}{5}.$$

7. Line integral of the form (3) on p. 414. In space. Here the path of integration C is a portion of an "exponential" helix, by Example 4, p. 383, of Sec. 9.5, with $a = 1$ and the third component replaced by e^t.

$$C : \mathbf{r}(t) = [\cos t, \quad \sin t, \quad e^t].$$

Note that t varying from 0 to 2π is consistent with the given endpoints $(1, 0, 1)$ and $(1, 0, e^{2\pi})$ of the path of integration, which you can verify by substitution.

For the integrand in (3), p. 414, we need the expression of $\mathbf{F}$ on C

$$\mathbf{F}(\mathbf{r}(t)) = [\cos^2 t, \quad \sin^2 t, \quad e^{2t}].$$

We also need the tangent vector of C

$$\mathbf{r}'(t) = [-\sin t, \quad \cos t, \quad 1].$$

Then the dot product is

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) &= [\cos^2 t, \quad \sin^2 t, \quad e^{2t}] \bullet [-\sin t, \quad \cos t, \quad 1] \\ &= -\cos^2 t \sin t + \sin^2 t \cos t + e^{3t}.\end{aligned}$$

By substitution from regular calculus with

$$u = \cos t, \qquad \frac{du}{dt} = -\sin t, \qquad du = -\sin t \, dt,$$

we get

$$\int \cos^2 t \sin t \, dt = -\int u^2 du = -\frac{u^3}{3} + \text{const} = -\frac{1}{3}\cos^3 t + \text{const}.$$

Similarly for $\sin^2 t \cos t$. We are ready to set up and solve the line integral:

$$\begin{aligned}\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} &= \int_0^{2\pi} (-\cos^2 t \sin t + \sin^2 t \cos t + e^{3t})\, dt \\ &= \left[\frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t + \frac{1}{3}e^{3t}\right]_0^{\pi/2} \\ &= \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot e^{6\pi} - \left(\frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1\right) = \frac{1}{3}(e^{6\pi} - 1).\end{aligned}$$

15. Line integral in space. Use (8) on p. 417. We have to integrate $\mathbf{F} = [y^2, \quad z^2, \quad x^2]$ over the helix $C : \mathbf{r}(t) = [3\cos t, \quad 3\sin t, \quad 2t]$ from $t = 0$ to $t = 4\pi$. The integrand in (8) is

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) &= [y^2(t), \quad z^2(t), \quad x^2(t)] \\ &= [9\sin^2 t, \quad 4t^2, \quad 9\cos^2 t].\end{aligned}$$

Then we integrate

$$I = \int_0^{4\pi} [9\sin^2 t, \quad 4t^2, \quad 9\cos^2 t]\, dt.$$

We have to integrate each component over t from 0 to 4π. We know from calculus (inside cover of textbook) that

$$\int \sin^2 t\, dt = \frac{1}{2}t - \frac{1}{4}\sin 2t + \text{const}; \qquad \int \cos^2 t\, dt = \frac{1}{2}t + \frac{1}{4}\sin 2t + \text{const}$$

so that

$$9\int_0^{4\pi} \sin^2 t\, dt = 9\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{4\pi} = 18\pi; \qquad 9\int_0^{4\pi} \cos^2 t\, dt = 18\pi.$$

Furthermore,

$$\int_0^{4\pi} 4t^2\, dt = \frac{4}{3}(4\pi)^3.$$

Thus we have

$$I = \left[18\pi, \quad \tfrac{4}{3}(4\pi)^3, \quad 18\pi\right].$$

19. Line integral of the form (8*), p. 417. In space. We have to integrate $f = xyz$ over the curve $C : \mathbf{r}(t) = [4t, \quad 3t^2, \quad 12t]$ from $t = -2$ to $t = 2$, that is, from $(x, y, z) = (-8, 12, -24)$ to $(x, y, z) = (8, 12, 24)$. The integrand is

$$f(\mathbf{r}(t)) = x(t)y(t)z(t) = 4t \cdot 3t^2 \cdot 12t = 144t^4$$

so that

$$\int_{-2}^{2} f(\mathbf{r}(t))\, dt = \int_{-2}^{2} 144t^4\, dt = 144\frac{t^5}{5}\bigg|_{-2}^{2} = \frac{144}{5}[2^5 - (-2)^5] = \frac{144}{5}\cdot 2^6 = 1843.2.$$

Sec. 10.2 Path Independence of Line Integrals

Again consider a line integral

$$\text{(A)} \qquad \int_C \mathbf{F} \cdot d\mathbf{r}$$

in a domain D. The question arises whether the *choice of a path* C (within a domain D) between two points A and B that are fixed affects the value of a line integral (A). Usually it does, as we saw in Sec. 10.1 (Theorem 2, p. 418 with example). Then (A) is *path dependent.* If it does not, then (A) is **path independent**.

Summary of Sec. 10.2

Equation (A) is path independent if and only if $\mathbf{F}$ is the gradient of some function f, called a *potential* of $\mathbf{F}$. This is shown in **Theorem 1**, p. 420, and applied to **Examples 1 and 2**, p. 421, and **Prob. 3**.

Equation (A) is path independent if and only if it is 0 for every closed path (**Theorem 2**, p. 421).

Equation (A) is path independent if and only if the differential form $F_1\,dx + F_2\,dy + F_3\,dz$ is exact (Theorem 3*, p. 423).

Also if (A) is path independent, then curl $\mathbf{F} = \mathbf{0}$ (**Theorem 3**, p. 423, **Example 3**, p. 424). If curl $\mathbf{F} = \mathbf{0}$ and D is simply connected, then (A) is path independent (Theorem 3, Prob. 3).

Path independence is highly desirable in applications of mechanics and elastic springs, as discussed on p. 419. The section summary shows that methods for checking path independence involve trying to find a potential or computing the curl [(1), p. 407 of Sec. 9.9].

Problem Set 10.2. Page 425

3. Path Independence. Theorem 1 on p. 420 and Example 1 on p. 421 give us a solution strategy. To solve the given integral, it would be best if we could find a potential f (if it exists) that relates to $\mathbf{F}$. Then Theorem 1 would guarantee path independence.

From the differential form under the integral

$$\tfrac{1}{2}\cos\tfrac{1}{2}x\cos 2y\,dx - 2\sin\tfrac{1}{2}x\sin 2y\,dy.$$

we have

$$\mathbf{F} = \left[\tfrac{1}{2}\cos\tfrac{1}{2}x\cos 2y, \quad -2\sin\tfrac{1}{2}x\sin 2y\right],$$

and if we can find a potential, then

$$\text{(B)} \qquad \mathbf{F} = \operatorname{grad} f = [f_x, \quad f_y],$$

where the indices denote partial derivatives. We try to find f. We have

$$f_x = \tfrac{1}{2}\cos\tfrac{1}{2}x\cos 2y$$

integrated with respect to x is

$$
\begin{aligned}
f &= \int \frac{1}{2} \cos \frac{1}{2}x \cdot \cos 2y \, dx = \frac{1}{2} \cos 2y \int \cos \frac{1}{2}x \, dx \\
&= \frac{1}{2} \cos 2y \cdot 2 \sin \frac{1}{2}x + g(y) \\
&= \cos 2y \cdot \sin \frac{1}{2}\, x + g(y).
\end{aligned}
$$

Similarly,

$$f_y = -2 \sin \tfrac{1}{2}x \sin 2y$$

integrated with respect to y is

$$
\begin{aligned}
f &= \int -2 \sin \frac{1}{2}x \sin 2y \, dy = -2 \sin \frac{1}{2}x \int \sin 2y \, dy \\
&= -2 \sin \frac{1}{2}x \cdot \left(-\frac{1}{2} \cos 2y\right) + h(x) \\
&= \sin \frac{1}{2}x \cdot \cos 2y + h(x).
\end{aligned}
$$

Comparing the two integrals, just obtained, allows us to chose g and h to be zero. Thus

$$f = \sin \tfrac{1}{2}x \cdot \cos 2y.$$

We see that f satisfies (B) and, by Theorem 1, we have path independence. Hence we can use (3), p. 420, to calculate the value of the desired integral. We insert the upper limit of integration to obtain

$$\sin \frac{1}{2}x \cdot \cos 2y \Big|_{x=\pi, y=0}^{2} = \sin \frac{\pi}{2} \cdot \cos (2 \cdot 0) = 1$$

and the lower limit to obtain

$$\sin \frac{1}{2}x \cdot \cos 2y \Big|_{x=\pi/2, y=\pi}^{2} = \sin \frac{\pi}{4} \cdot \cos 2\pi = \frac{1}{\sqrt{2}}.$$

Together the answer is $1 - \frac{1}{\sqrt{2}}$.

A more complicated example, where g and h are *not* constant, is illustrated in Example 2 on p. 421 and also in **Prob. 43** on p. 402 of Sec. 9.7 and solved on p. 163 in this Manual.

Note the following. We were able to choose (3) only after we had shown path independence. Second, how much freedom did we have in choosing g and h? Since the two expressions must be equal, we must have $g(y) = h(x) = \text{const}$. And this arbitrary constant drops out in the difference in (3), giving us a unique value of the integral.

Considerations. How far would we have come, using Theorem 3, p. 423? Since $\mathbf{F}$ is independent of z, (6) in Theorem 3 can be replaced by (6″). Thus we compute

$$\frac{\partial \mathbf{F}_2}{\partial x} = \frac{\partial}{\partial x}\left(-2 \sin \frac{1}{2}x \sin 2y\right) = -2 \cdot \frac{1}{2} \cos \frac{1}{2}x \sin 2y = -\cos \frac{1}{2}x \cdot \sin 2y$$

and

$$\frac{\partial \mathbf{F}_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2}\cos\frac{1}{2}x\cos 2y\right) = (-2\sin 2y)\left(\frac{1}{2}\cos\frac{1}{2}x\right) = -\sin 2y \cdot \cos\frac{1}{2}x.$$

Hence

$$\frac{\partial \mathbf{F}_2}{\partial x} = \frac{\partial \mathbf{F}_1}{\partial y},$$

which means that curl $\mathbf{F} = \mathbf{0}$. By Theorem 3, we conclude path independence in a simply connected domain, e.g., in the whole xy-plane. We could now integrate to obtain, say,

$$f = \int -2\sin\frac{1}{2}x\sin 2y\,dy = \sin\frac{1}{2}x \cdot \cos 2y + h(x)$$

and use

$$f_x = \frac{\partial}{\partial x}\left(\sin\frac{1}{2}x \cdot \cos 2y\right) + h'(x) = \frac{1}{2}\cos\frac{1}{2}x\cos 2y + h'(x) = \frac{1}{2}\cos\frac{1}{2}x\cos 2y$$

to get $h'(x) = 0$, hence $h(x) = \text{const}$ as before.

15. Path Independence? From the given differential form

$$x^2y\,dx - 4xy^2dy + 8z^2x\,dz$$

we write

$$\mathbf{F} = [x^2y, \quad -4xy^2, \quad 8z^2x].$$

Then we have to compute

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & -4xy & 8z^2x \end{vmatrix}.$$

We have

$$\begin{aligned} &\frac{\partial}{\partial y}(8z^2x) = 0; && \frac{\partial}{\partial z}(-4xy^2) = 0; \\ &\frac{\partial}{\partial z}(x^2y) = 0; && \frac{\partial}{\partial x}(8z^2x) = 8z^2; \\ &\frac{\partial}{\partial x}(-4xy^2) = -4y^2; && \frac{\partial}{\partial y}(x^2y) = x^2; \end{aligned}$$

so that

$$\begin{aligned} \text{curl}\,\mathbf{F} &= (0-0)\mathbf{i} + (0 - 8z^2)\mathbf{j} + (-4y^2 - x^2)\mathbf{k} \\ &= [0, \quad -8z^2, \quad -4y^2 - x^2]. \end{aligned}$$

The vector obtained is *not* the zero vector in general, indeed

$$-4y^2 - x^2 \neq 0, \qquad \text{i.e.,} \quad x^2 \neq -4y^2.$$

This means that the differential form is not exact. Hence by Theorem 3, p. 423, we have **path dependence** in any domain.

Sec. 10.3 Calculus Review: Double Integrals. *Optional*

In Sec. 10.4 we shall transform line integrals into double integrals. In Secs. 10.6 and 10.9 we shall transform surface integrals also into double integrals. **Thus you have to be able to readily set up and solve double integrals.**

To check whether you remember how to solve double integrals, do the following problem: Evaluate $\iint\limits_R x^3\,dx\,dy$ where R is the region in the first quadrant (i.e., $x \geq 0$, $y \geq 0$) that is bounded by and lies between $y = x^2$ and the line $y = x$ with $0 \leq x \leq 1$. Sketch the area and then solve the integral. [*Please close this Student Solutions Manual (!) and do it by paper and pencil or type on your computer without looking or using a CAS and then compare the result on p. 199 of this chapter with your solution*].

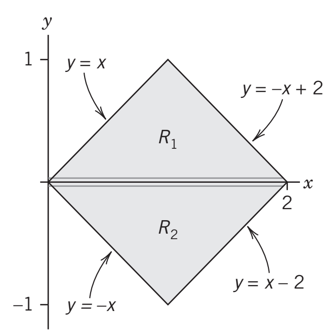
If you got the correct answer, great. If not, then you certainly need to review and practice double integrals.

More details on Example 1, p. 430. Change of variables in double integral.

Method I: Solution without transformation. It is instructional to review how to set up the region of integration. First, we have to determine the boundaries of the tilted blue square R in Fig. 232, p. 430. It is formed by the intersection of four straight lines:

$$\text{(A) } y = x, \qquad \text{(B) } y = -x + 2,$$
$$\text{(C) } y = x - 2, \qquad \text{(D) } y = -x.$$

To set up the problem we break the region R into R_1 and R_2 as shown in the figure below.



Sec. 10.3 Example 1. *Method I.* Direct solution of double integral by breaking region R into regions R_1 and R_2

For R_1: We integrate first over x and then over y. Accordingly, we express (A) and (B) in terms of x. Equations (A) and (B) are given in the form of (3), p. 427, but we need them in the form of (4) on p. 428. We solve for x.

$$R_1: \quad \text{(A*) } x = y, \qquad \text{(B*) } x = -y + 2.$$

This gives us that $y \leq x \leq -y + 2$. Also $0 \leq y \leq 1$. Hence

$$\iint\limits_{R_1} (x^2 + y^2)\,dx\,dy = \int_0^1 \int_y^{-y+2} (x^2 + y^2)\,dx\,dy = \int_{y=0}^1 \int_{x=y}^{-y+2} x^2\,dx\,dy + \int_{y=0}^1 \int_{x=y}^{-y+2} y^2\,dx\,dy.$$

We solve

$$\begin{aligned}\int_{y=0}^{1}\int_{x=y}^{-y+2} x^2\,dx\,dy &= \int_0^1 \left[\frac{x^3}{3}\right]_y^{-y+2} dy \\ &= \int_0^1 \frac{1}{3}[(-y+2)^3 - y^3]\,dy \\ &= \frac{1}{3}\int_0^1 (-y+2)^3\,dy - \frac{1}{3}\int_0^1 y^3\,dy = \frac{1}{3}\cdot\frac{15}{4} + \frac{1}{3}\cdot\frac{1}{4} = \frac{7}{6},\end{aligned}$$

and

$$\begin{aligned}\int_{y=0}^{1}\int_{x=y}^{-y+2} y^2\,dx\,dy &= \int_{y=0}^{1} y^2\left(\int_{x=y}^{-y+2} dx\right)dy \\ &= \int_{y=0}^{1} y^2\big[x\big]_y^{-y+2}dy = \int_0^1 (-2y^3+2y^2)\,dy = -2\cdot\frac{1}{4} + 2\cdot\frac{1}{3} = \frac{1}{6}\end{aligned}$$

so that the desired integral, over R_1, has the value

$$\tfrac{7}{6} + \tfrac{1}{6} = \tfrac{8}{6} = \tfrac{4}{3}.$$

For R_2: Since $f(x, y) = f(x, -y)$ for $f(x, y) = x^2 + y^2$, and by symmetry, the double integral over R_2 has the same value as that over R_1. Hence the double integral over the total area R has the value of $2\cdot\frac{4}{3} = \frac{8}{3}$—giving us the final answer—or similarly, we can continue and also set up the integral for R_2 (verify for practice!):

$$\iint_{R_2} (x^2+y^2)\,dx\,dy = \int_{-1}^{0}\int_{-y}^{y+2} (x^2+y^2)\,dx\,dy = \frac{4}{3},$$

and add up the two double integrals for $R_1$and R_2 to again get $\frac{8}{3}$.

Method II: With transformation. More details on the solution of the textbook. The problem is unusual in that the suggested transformation is governed by the region of integration rather than by the integrand. Equations (A), (B), (C), and (D) can be written as

(A**) $\qquad x - y = 0,$

(B**) $\qquad x + y = 2,$

(C**) $\qquad x - y = 2,$

(D**) $\qquad x + y = 0.$

This lends itself to set (S1) $x + y = u$ and (S2) $x - y = v$. Then (A**), (B**), (C**), and (D**) become

$$v = 0, \qquad u = 2, \qquad v = 2, \qquad u = 0.$$

This amounts to a rotation of axes so that the square is now parallel to the uv-axes and we can set up the problem as one (!) double integral which is easier to solve. Its limits of integration are $0 \le u \le 2$ and $0 \le v \le 2$. Look at the tilted blue square in Fig. 232, p. 430, in the textbook. Furthermore, (S1) gives us $x = u - y$. From (S2) we have $-y = v - x$, which we can substitute into $x = u - y = u + v - x$, so that $2x = u + v$ and

(E) $$x = \tfrac{1}{2}(u+v).$$

Also $-y = v - x$, so $y = -v + x = -v + \frac{1}{2}(u+v) = -v + \frac{1}{2}u + \frac{1}{2}v$, so that

(F)
$$y = \tfrac{1}{2}(u - v).$$

The change of variables requires by (6), p. 429, the Jacobian J. For purpose of uniqueness (see p. 430), we use $|J|$. We get

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} \qquad \text{so that} \qquad |J| = \frac{1}{2}.$$

Note that for J we computed the partials

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left[\frac{1}{2}(u+v)\right] = \frac{\partial}{\partial u}\left(\frac{1}{2}u + \frac{1}{2}v\right) = \frac{1}{2} + 0 = \frac{1}{2}, \quad \text{etc.}$$

Substituting (E) and (F) into $x^2 + y^2$ together with $|J|$ gives the double integral in u and v of Example 1, p. 430.

You may ask yourself why we presented two ways to solve the problem. Two reasons:

1. If you cannot think of an elegant way, an inelegant way may also work.
2. More practice in setting up double integrals.

Problem. Set 10.3. Page 432

3. Double integral. Perhaps the only difficulty in this and similar problems is to figure out the region of integration if it is given "implicitly" as in this problem. We first integrate in the horizontal (x-direction) from the straight line $x = -y$ to the straight line $x = y$. Then we integrate in the vertical direction (y-direction) from $y = 0$ to $y = 3$. Thus the region lies between by the lines $y = -x$ and $y = x$ above the x-axis and below the horizontal line $y = 3$ and includes the boundaries. Sketch it and you see that the region of integration is the triangle with vertices $(-3, 3)$, $(3, 3)$, and $(0, 0)$. We obtain

$$\begin{aligned}
\int_0^3 \int_{-y}^{y} [(x^2 + y^2)\, dx]\, dy &= \int_{y=0}^{3} \int_{x=-y}^{y} \left[x^2 dx\right] dy + \int_{y=0}^{3} \int_{x=-y}^{y} \left[y^2 dx\right] dy \\
&= \int_{y=0}^{3} \int_{x=-y}^{y} \left[x^2 dx\right] dy + \int_{y=0}^{3} \left(y^2 \int_{x=-y}^{y} dx\right) dy \\
&= \int_{y=0}^{3} \frac{x^3}{3}\bigg|_{x=-y}^{y} dy + \int_{y=0}^{3} y^2 \left[x\right]_{x=-y}^{y} dy \\
&= \int_{y=0}^{3} \left[\frac{y^3}{3} - \left(-\frac{y^3}{3}\right) + y^2(y - (-y))\right] dy \\
&= \int_0^3 \left[\frac{y^3}{3} + \frac{y^3}{3} + y^3 + y^3\right] dy = \frac{8}{3}\int_0^3 y^3\, dy \\
&= \frac{8}{3}\left[\frac{y^4}{4}\right]_{y=0}^{y=3} = \frac{8}{3} \cdot \left(\frac{3^4}{4} + 0\right) = 2 \cdot 3^3 = 54.
\end{aligned}$$

13. Center of gravity. The main task is to correctly get the region of integration. The given region R is a triangle. We see that, horizontally, we have to integrate from $x = 0$ to $x = b$. Now comes some thinking. We have to first integrate y from 0 to the largest side of the triangle, the hypothenuse. Now, as we go b *units* to the right, we have to go h *units* up. Thus if we go 1 *unit* to the right we have to go h/b *units* up, so that, the slope of the desired line is $\widetilde{m} = h/b$. Since the line goes through the origin, its y-intercept is 0, so the equation is

$$y = \frac{h}{b}x.$$

Thus, vertically, we have to integrate from $y = 0$ to $y = (h/b)x$. Furthermore, the total mass of the triangle is $M = \frac{1}{2}bh$. Hence we calculate, using formulas on p. 429,

$$\begin{aligned}\bar{x} = \frac{1}{M}\int_{x=0}^{b}\left(\int_{y=0}^{hx/b} x\,dy\right)dx &= \frac{1}{M}\int_{x=0}^{b} x \cdot \frac{hx}{b}\,dx \\ &= \frac{2}{bh}\cdot\frac{h}{b}\cdot\frac{x^3}{3}\bigg|_0^b = \frac{2b}{3}.\end{aligned}$$

Similarly,

$$\begin{aligned}\bar{y} = \frac{1}{M}\int_{x=0}^{b}\left(\int_{y=0}^{hx/b} y\,dy\right)dx &= \frac{1}{M}\int_{x=0}^{b} \frac{1}{2}\left(\frac{hx}{b}\right)^2 dx \\ &= \frac{2}{bh}\cdot\frac{1}{2}\left(\frac{h}{b}\right)^2\frac{x^3}{3}\bigg|_0^b = \frac{1}{3}h.\end{aligned}$$

We have that $\bar{x}$ is independent of h and $\bar{y}$ is independent of b. Did you notice it? Can you explain it physically?

17. Moments of inertia. From the formulas on p. 429 we have

$$\begin{aligned}I_x &= \iint_R y^2 f(x,y)\,dx\,dy = \iint_R y^2 \cdot 1\,dx\,dy \\ &= \int_{x=0}^{b}\left(\int_{y=0}^{hx/b} y^2 dy\right)dx = \int_{x=0}^{b}\left(\left[\frac{y^3}{3}\right]_{y=0}^{hx/b}\right)dx \\ &= \int_{x=0}^{b}\frac{1}{3}\cdot\frac{h^3x^3}{b^3}\,dx = \frac{1}{3}\frac{h^3}{b^3}\int_{x=0}^{b} x^3\,dx \\ &= \frac{1}{3}\frac{h^3}{b^3}\left[\frac{x^4}{4}\right]_0^b = \frac{1}{12}bh^3.\end{aligned}$$

Similarly,

$$\begin{aligned}I_y &= \iint_R x^2 f(x,y)\,dx\,dy = \iint_R x^2 \cdot 1\,dx\,dy \\ &= \int_{x=0}^{b}\left(\int_{y=0}^{hx/b} x^2 dy\right)dx = \int_{x=0}^{b} x^2\frac{hx}{b}\,dx = \frac{1}{4}hb^3.\end{aligned}$$

Adding the two moments together gives the polar moment of inertia I_0 (defined on p. 429):

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2) f(x, y)\, dx\, dy = \frac{1}{12} bh^3 + \frac{1}{4} hb^3 = \frac{bh}{12}(h^2 + 3b^2).$$

Sec. 10.4 Green's Theorem in the Plane

Green's theorem in the plane (Theorem 1, p. 433) transforms double integrals over a region R in the xy-plane into line integrals over the boundary curve C of R and, conversely, line integrals into double integrals. These transformations are of practical and theoretical interest and are used—depending on the purpose—in both directions. For example, **Prob. 3** transforms a line integral into a simpler double integral.

Formula (9), p. 437, in the text and formulas (10)–(12) in the problem set on p. 438 are remarkable consequences of Green's theorem in the plane. For instance, if w is harmonic ($\nabla^2 w = 0$), then its normal derivative over a closed curve is zero, by (9). For other functions, (9) may simplify the evaluation of integrals of the normal derivative (**Prob. 13**). Such integrals of the normal derivative occur in fluid flow in connection with the flux of a fluid through a surface.

We shall also need Green's theorem in the plane in Sec. 10.9.

Problem Set 10.4. Page 438

3. Transformation of a line integral into a double integral by Green's theorem in the plane. In **Probs. 1–10** we use Green's theorem in the plane to transform line integrals over a boundary curve C of R into double integrals over a region R in the xy-plane. Note that it would be much more involved if we solved the line integrals directly. Given

$$\mathbf{F} = [F_1, \quad F_2] = [x^2 e^y, \quad y^2 e^x].$$

We use Green's theorem in the plane, that is (1), p. 433, as follows:

$$\text{(GT)} \qquad \oint_C (F_1 + F_2)\, dx\, dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_2}{\partial y} \right) dx\, dy.$$

We need

$$\text{(A)} \qquad \frac{\partial F_2}{\partial x} - \frac{\partial F_2}{\partial y} = (F_2)_x - (F_1)_y = y^2 e^x - x^2 e^y.$$

You should sketch the given rectangle, which is the region of integration R in the double integral on the left-hand side of (GT). From your sketch you see that we have to integrate over x from 0 to 2 and over y from 0 to 3. We have

$$\text{(B)} \qquad \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_2}{\partial y} \right) dx\, dy = \int_{y=0}^{3} \left[\int_{x=0}^{2} (y^2 e^x - x^2 e^y)\, dx \right] dy.$$

We first integrate (A) over x, which gives us the indefinite integral

$$\int (y^2 e^x - x^2 e^y)\, dx = y^2 e^x - \frac{x^3}{3} e^y.$$

From this we obtain the value of the definite integral by substituting the upper limit $x = 2$ and then the lower limit $x = 0$ and taking the difference

$$\left(y^2e^2 - \tfrac{8}{3}e^y\right) - (y^2 \cdot 1 - 0) = y^2e^2 - \tfrac{8}{3}e^y - y^2,$$

so that

$$\int_{x=0}^{2} (y^2e^x - x^2e^y)\,dx = y^2e^2 - \tfrac{8}{3}e^y - y^2.$$

Next we integrate this over y and obtain

$$\int \left(y^2e^2 - \frac{8}{3}e^y - y^2\right) dy = \frac{1}{3}y^3e^2 - \frac{8}{3}e^y - \frac{1}{3}y^3.$$

We substitute the upper limit $y = 3$ and the lower limit $y = 0$, and take the difference of the two expressions obtained and get the corresponding definite integral:

$$\int_{y=0}^{3} \left(y^2e^2 - \frac{8}{3}e^y - y^2\right) dy = \left(\frac{27}{3}e^2 - \frac{8}{3}e^3 - \frac{27}{3}\right) - \left(0 - \frac{8}{3} \cdot 1 - 0\right) = 9e^2 - \frac{8}{3}e^3 - 9 + \frac{8}{3}.$$

Hence

$$\int_{y=0}^{3} \left[\int_{x=0}^{2} (y^2e^x - x^2e^y)\,dx\right] dy = 9e^2 - \frac{8}{3}e^3 - 9 + \frac{8}{3}.$$

13. Integral of the normal derivative. Use of Green's theorem in the plane. We use (9), p. 437, to simplify the evaluation of an integral of the normal derivative by transforming it into a double integral of the Laplacian of a function. Such integrals of the normal derivative are used in the flux of a fluid through a surface and in other applications.

For the given $w = \cosh x$, we obtain, by two partial derivatives, that the Laplacian of w is $\nabla^2 w = w_{xx} = \cosh x$. You should sketch the region of integration of the double integral on the left-hand side of (9), p. 437. It is a triangle with a 90° angle. A similar reasoning as in Prob. 13 of Sec. 10.3 (see solution before) gives us that the hypothenuse of the triangle has the representation

$$y = \tfrac{1}{2}x \qquad \text{which implies that} \qquad x = 2y.$$

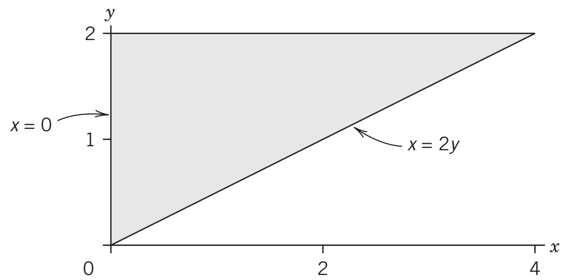
Hence we integrate the function $\cosh x$ from $x = 0$ horizontally to $x = 2y$. The result of that integral is integrated vertically from $y = 0$ to $y = 2$. We have by (9)

$$\begin{aligned}
\oint_C \frac{\partial w}{\partial n}\,ds &= \iint_R \nabla^2 w\,dx\,dy \\
&= \int_{y=0}^{2} \left(\int_{x=0}^{2y} \cosh x\,dx\right) dy = \int_{y=0}^{2} \sinh 2y\,dy \\
&= \left[\frac{1}{2}\cosh 2y\right]_0^2 = \frac{1}{2}(\cosh 4 - 1).
\end{aligned}$$

Note that the first equality required integration over x:

$$\left(\int_{x=0}^{2y} \cosh x\,dx\right) = [\sinh x]_0^{2y} = \sinh 2y - \sinh 0 = \sinh 2y,$$

which gave us the integrand $\sinh 2y$ to be integrated over y (from $y = 0$ to $y = 2$).



Sec. 10.4 Prob 13. Region of integration in double integral

19. Laplace's equation. Here we use (12), p. 438, another consequence of Green's theorem. From $w = e^x \sin y$ we find, by differentiation, that the Laplacian of w is

$$\nabla^2 w = w_{xx} + w_{yy} = e^x \sin y + e^x(-\sin y) = 0.$$

Since the Laplacian is zero, we can apply formula (12) (in Prob. 18, p. 438) involving

$$w_x = e^x \sin y, \qquad w_y = e^x \cos y.$$

We calculate from this

$$w_x^2 + w_y^2 = e^{2x} \sin^2 y + e^{2x} \cos^2 y = e^{2x}.$$

Using (12),

$$\begin{aligned}\oint_C w \frac{\partial w}{\partial n}\, ds &= \iint_R \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] dx\, dy \\ &= \int_{y=0}^{5} \left(\int_{x=0}^{2} e^{2x}\, dx \right) dy.\end{aligned}$$

Integration over x (chain rule!) from $x = 0$ to $x = 2$ yields

$$\int_{x=0}^{2} e^{2x}\, dx = \left[\frac{1}{2} e^{2x} \right]_0^2 = \frac{1}{2}(e^4 - 1).$$

Integration over y from 0 to 5 (according to the given square) introduces a factor 5:

$$\int_{y=0}^{5} \frac{1}{2}(e^4 - 1)\, dx = \frac{1}{2}(e^4 - 1) \int_{y=0}^{5} dx = 5 \cdot \frac{1}{2}(e^4 - 1).$$

Sec. 10.5 Surfaces for Surface Integrals

Section 10.5 is a companion to Sec. 9.5, with the focus on three dimensions. We consider surfaces (in space), along with tangent planes and surface normals, only to the extent we shall need them in connection with surface integrals, as suggested by the title. *Try to gain a good understanding of parametric representations of surfaces* by carefully studying the standard examples in the text (cylinder, p. 440; sphere, p. 440; and cone, p. 441). You may want to continue building your table of parametric representations whose purpose was discussed on p. 156 of this Solution Manual. If you do not have a table, start one.

Problem Set 10.5. Page 442

3. Parametric surface representations have the advantage that the components x, y, and z of the position vector $\mathbf{r}$ play the same role in the sense that none of them is an independent variable (as it is the case when we use $z = f(x, y)$), but all three are functions of two variables (**parameters**) u and v (we need two of them because a surface is two-dimensional). Thus, in the present problem,

$$\mathbf{r}(u, v) = [x(u, v), \quad y(u, v), \quad z(u, v)] = [u \cos v, \quad u \sin v, \quad cu].$$

In components,

$$\text{(A)} \qquad x = u \cos v, \qquad y = u \sin v, \qquad z = cu \quad (c \text{ constant}).$$

If cos and sin occur, we can often use $\cos^2 v + \sin^2 v = 1$. At present,

$$x^2 + y^2 = u^2(\cos^2 v + \sin^2 v) = u^2.$$

From this and $z = cu$ we get

$$z = c\sqrt{x^2 + y^2}.$$

This is a representation of the cone of the form $z = f(x, y)$.

If we set $u = \text{const}$, we see that $z = \text{const}$, so these curves are the intersections of the cone with horizontal planes $u = \text{const}$. They are circles.

If we set $v = \text{const}$, then $y/x = \tan v = \text{const}$ (since u drops out in (A)). Hence $y = kx$, where $k = \tan v = \text{const}$. These are straight lines through the origin in the xy-plane, hence they are planes, through the z-axis in space, which intersect the cone along straight lines.

To find a surface normal, we first have to calculate the partial derivatives of $\mathbf{r}$,

$$\begin{aligned}\mathbf{r}_u &= [\cos v, \quad \sin v, \quad c],\\ \mathbf{r}_v &= [-u \sin v, \quad u \cos v, \quad 0],\end{aligned}$$

and then form their cross product $\mathbf{N}$ because this cross product is perpendicular to the two vectors, which span the tangent plane, so that $\mathbf{N}$, in fact, is a normal vector. We obtain

$$\begin{aligned}\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & c \\ -u \sin v & u \cos v & 0 \end{vmatrix}\\ &= \mathbf{i}\begin{vmatrix} \sin v & c \\ u \cos v & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \cos v & c \\ -u \sin v & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \cos v & \sin v \\ -u \sin v & \cos v \end{vmatrix}\\ &= [-cu \cos v, \quad -cu \sin v, \quad u].\end{aligned}$$

In this calculation, the third component resulted from simplification:

$$(\cos v)u \cos v - (\sin v)(-u \sin v) = u(\cos^2 v + \sin^2 v) = u.$$

13. Derivation of parametric representations. This derivation from a nonparametric representation is generally simpler than the inverse process. For example, the plane $4x - 2y + 10z = 16$, like any other surface, can be given by many different parametric representations.

Set $x = u$, $y = v$ to obtain

$$10z = 16 - 4x + 2y = 16 - 4u + 2v,$$

so that

$$z = f(x, y) = f(u, v) = 1.6 - 0.4u + 0.2v$$

and

$$\mathbf{r} = r(u, v) = [u, v, f(u, v)] = [u, \quad v, \quad 1.6 - 0.4u + 0.2v].$$

If, in our illustration, we wish to get rid of the fractions in z, we set $x = 10u$, $y = 10v$ and write

$$\mathbf{r} = [10u, \quad 10v, \quad 1.6 - 4u + 2v].$$

A normal vector $\mathbf{N}$ of such a representation $\mathbf{r}(u, v) = [u, \quad v, \quad f(u, v)]$ is now obtained by first calculating the partial derivatives

$$\mathbf{r}_u = [1, \quad 0, \quad f_u],$$
$$\mathbf{r}_v = [0, \quad 1, \quad f_v]$$

and then their cross product (same type of calculation as in Prob. 3, before)

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = [-f_u, \quad -f_v, \quad 1].$$

This is precisely (6) on p. 443. You should fill in the details of the derivation.

Sec. 10.6 Surface Integrals

Overview of Sec. 10.6

- Definition of surface integral over an oriented surface (3)–(5), pp. 443–444
- Flux: Example 1, pp. 444–445
- Surface orientation, practical and theoretical aspects: pp. 445–447
- Surface integral without regard to orientation: pp. 448–452 (and **Prob. 15**)
- Surface area: p. 450

Problem Set 10.6. Page 450

1. Surface integral over a plane in space. The surface S is given by

$$\mathbf{r}(u, v) = [u, \quad v, \quad 3u - 2v].$$

Hence $x = u$, $y = v$, $z = 3u - 2v = 3x - 2y$. This shows that the given surface is a plane in space. The region of integration is a rectangle: u varies from 0 to 1.5 and v from -2 to 2. Since $x = u$, $y = v$, this is the same rectangle in the xy-plane.

In (3), p. 443, on the right, we need the normal vector $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$. Taking, for each component, the partial with respect to u we get

$$\mathbf{r}_u = [1, \quad 0, \quad 3].$$

Similarly

$$\mathbf{r}_v = [0, \quad 1, \quad -2].$$

Then by (2**), p. 370,

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$
$$= -3\mathbf{i} - (-2)\mathbf{j} + \mathbf{k} = [-3, \quad 2, \quad 1].$$

Next calculate $\mathbf{F}$ on the surface S. We do this by substituting the components of $\mathbf{r}$ into $\mathbf{F}$. This gives us (with $x = u$, $y = v$):

$$\mathbf{F} = [-x^2, \quad y^2, \quad 0] = [-u^2, \quad v^2, \quad 0].$$

Hence the dot product on the right-hand side of (3) is

$$\mathbf{F} \bullet \mathbf{N} = [-u^2, \quad v^2, \quad 0] \bullet [-3, \quad 2, \quad 1] = 3u^2 + 2v^2.$$

The following is quite interesting. Since $\mathbf{N}$ is a cross product, $\mathbf{F} \bullet \mathbf{N}$ is a scalar triple product (see (10), p. 373 in Sec. 9.3) and is thus given by the determinant

$$(\mathbf{F} \quad \mathbf{r}_u \quad \mathbf{r}_v) = \begin{vmatrix} -u^2 & v^2 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & -2 \end{vmatrix} = -u^2 \begin{vmatrix} 0 & 3 \\ 1 & -2 \end{vmatrix} - v^2 \begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$
$$= 3u^2 + 2v^2,$$

agreeing with our previous calculation. Note that, in this way, we have done two steps in one!

Next we need to calculate

$$\iint_S \mathbf{F} \bullet \mathbf{N}\, d\mathbf{A} = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \bullet \mathbf{N}(u, v)\, du\, dv$$
$$= \int_{-2}^{2} \int_{0}^{1.5} (3u^2 + 2v^2)\, du\, dv = \int_{-2}^{2} \int_{0}^{1.5} 3u^2\, du\, dv + \int_{-2}^{2} \int_{0}^{1.5} 2v^2\, du\, dv$$
$$= \int_{-2}^{2} \big[u^3\big]_0^{1.5} dv + \int_{-2}^{2} 2v^2 \big[u\big]_0^{1.5}\, dv$$
$$= \int_{-2}^{2} (1.5)^3\, dv + \int_{-2}^{2} 2v^2 (1.5)\, dv$$
$$= (1.5)^3 \int_{-2}^{2} dv + 3 \int_{-2}^{2} v^2\, dv = (1.5)^3 \big[v\big]_{-2}^{2} + \big[v^3\big]_{-2}^{2}$$
$$= (1.5)^3 \cdot 4 + 2^3 + 2^3 = 13.5 + 16 = 29.5.$$

15. Surface integrals of the form (6), p. 448. We are given that $G = (1 + 9xz)^{3/2}$, $S : \mathbf{r} = [u, \quad v, \quad u^3]$, $0 \le u \le 1$, $-2 \le v \le 2$. Formula (6) gives us a surface integral without orientation, that is,

$$\iint_R \mathbf{G}(\mathbf{r})\, d\mathrm{A} = \iint_R \mathbf{G}(\mathbf{r}(u, v)) |\mathbf{N}(u, v)|\, du\, dv. \tag{6}$$

We set up the double integral on the right-hand side of (6). From the given parametrization of the surface S, we see that $x = u$ and $z = u^3$ and substituting this into G we obtain

$$\mathbf{G}(\mathbf{r}(u, v)) = (1 + 9uu^3)^{3/2} = (1 + 9u^4)^{3/2}.$$

We also need

$$\mathbf{r}_u = [1, \quad 0, \quad 3u^2]; \qquad \mathbf{r}_v = [0, \quad 1, \quad 0].$$

Then

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3u^2 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 3u^2 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3u^2 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k}$$

$$= -3u^2\mathbf{i} + \mathbf{k} = [-3u^2, \quad 0, \quad 1],$$

so that

$$|\mathbf{N}|^2 = \mathbf{N} \cdot \mathbf{N} = [-3u^2, \quad 0, \quad 1] \cdot [-3u^2, \quad 0, \quad 1] = (-3u^2)^2 + 0^2 + 1^2 = 9u^4 + 1.$$

Hence

$$|\mathbf{N}| = (9u^4 + 1)^{1/2}.$$

We are ready to set up the integral

$$\begin{aligned}
\iint_R \mathbf{G}(\mathbf{r}(u, v))\, |\mathbf{N}(u, v)|\, du\, dv &= \int_{v=-2}^{2} \int_{u=0}^{1} (1 + 9u^4)^{3/2}\, (9u^4 + 1)^{1/2}\, du\, dv \\
&= \int_{v=-2}^{2} \int_{u=0}^{1} (1 + 9u^4)^2\, du\, dv \\
&= \int_{v=-2}^{2} \int_{u=0}^{1} (1 + 18u^4 + 81u^8)\, du\, dv \\
&= \int_{v=-2}^{2} \left(\left[1 + 18\frac{u^5}{5} + 81\frac{u^9}{9} \right]_{u=0}^{1} \right) dv \\
&= \int_{v=-2}^{2} \left(1 + \frac{18}{5} + 9 \right) dv \\
&= \int_{v=-2}^{2} \frac{68}{5}\, dv = \frac{68}{5} v \Big|_{-2}^{2} = \frac{68}{5} \cdot 2 + \frac{68}{5} \cdot 2 = \frac{272}{5} = 54.4.
\end{aligned}$$

23. Applications. Moment of inertia of a lamina. Since the axis B is the z-axis, we know from Prob. 20, p. 451, that the moment of inertia of lamina S of density $\sigma = 1$ is

$$I_B = I_z = \iint_S (x^2 + y^2)\sigma\, dA = \iint_S (x^2 + y^2)\, dA. \tag{A}$$

We are given that $S : x^2 + y^2 = z^2$ so that S is a circular cone by Example 3, p. 441. This example also gives us a parametrization of S as

$$S : \mathbf{r} = [u \cos v, \quad u \sin v, \quad u], \qquad 0 \le u \le h, \qquad 0 \le v \le 2\pi.$$

This representation is convenient because, with $x = u \cos v$ and $y = u \sin v$, we have $z^2 = x^2 + y^2 = (u \cos v)^2 + (u \sin v)^2 = u^2(\cos^2 v + \sin^2 v) = u^2 \cdot 1$ so that, indeed, $z = u$ (corresponding to the third component of S). Furthermore we are given that $0 \le z \le h$ and hence u has the same range. Also $0 \le v \le 2\pi$ because it prescribes a circle. Using (6), p. 448, we need $|\mathbf{N}(u, v)|$. As before

$$\mathbf{N}(u, v) = \mathbf{r}_u \times r_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = -u \cos v\mathbf{i} - u \sin v\mathbf{j} + u\mathbf{k}.$$

Hence

$$\begin{aligned} |\mathbf{N}(u, v)| &= \sqrt{[-u \cos v, \quad -u \sin v, \quad u] \bullet [-u \cos v, \quad -u \sin v, \quad u]} \\ &= \sqrt{u^2(\cos^2 v + \sin^2 v) + u^2} = \sqrt{2}u. \end{aligned}$$

Putting it together by (A) and (6), p. 448,

$$\begin{aligned} I_B &= \iint_S (x^2 + y^2)\, dA = \iint_R \mathbf{G}(\mathbf{r}(u, v))|\mathbf{N}(u, v)|\, du\, dv \\ &= \iint_R u^2\sqrt{2}u\, du\, dv = \int_0^{2\pi} \left(\int_0^h u^2\sqrt{2}u\, du \right) dv. \end{aligned}$$

This evaluates as follows. The integral over u evaluates to

$$\left. \frac{\sqrt{2}u^4}{4} \right|_{u=0}^{h} = \frac{\sqrt{2}h^4}{4}.$$

Integrating the term just obtained over v, substituting the upper and lower limits, and simplifying gives us the final answer

$$\frac{\sqrt{2}h^4}{4} \cdot \big[v\big]_0^{2\pi} = \frac{1}{\sqrt{2}}\pi h^4,$$

which agrees with the answer on p. A26.

Sec. 10.7 Triple Integrals. Divergence Theorem of Gauss

We continue the main topic of transformation of different types of integrals. Recall the definition of divergence in Sec. 9.8 (div **F**, see (1), p. 402), which lends the name to this important theorem. The **Divergence Theorem** (Theorem 1, p. 453) allows us to transform triple integrals into surface integrals over the boundary surface of a region in space, and conversely surface integrals into triple integrals. **Example 2**, p. 456, and **Prob. 17** show us how to use the Divergence Theorem.

The proof of the divergence theorem hinges on proving (3)–(5), p. 454. First we prove (5) on pp. 454–455: We evaluate the integral over z by integrating over the projection of the region in the xy-plane (see the right side of (7), p. 455), thereby obtaining the surface integral in (5). Then the same idea is applied to (3) with x instead of z, and then to (4) with y instead of z.

More applications are in Sec. 10.8.

More details on Example 2, p. 456. Verification of Divergence Theorem.
(a) By Theorem 1 [(2), p. 453], we know that

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \iiint_T \operatorname{div} \mathbf{F}\, dV. \tag{2}$$

Here

$$\mathbf{F} = [F_1, \quad F_2, \quad F_3] = [7x, \quad 0, \quad -z],$$

so that by (1), p. 402,

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 7 + 0 - 1 = 6.$$

Thus the right-hand side of (2) becomes

$$\iiint_T \operatorname{div} \mathbf{F}\, dV = 6 \iiint_T dV = 6 \cdot [\text{volume of given sphere of radius 2}] = 6 \cdot \frac{4}{3}\pi \cdot 2^3 = 64\pi.$$

Computing the volume of a sphere by a triple integral. To actually get the volume of this sphere, we start with the parametrization

$$x = r\cos v \cos u, \qquad r \cos v \sin u, \qquad z = r \sin v$$

so that

$$\mathbf{r} = [r\cos v \cos u, \qquad y = r\cos v \sin u, \qquad r \sin v].$$

Then the **volume element** dV is $J\ dr\, du\, dv$. The Jacobian J is

$$J = \frac{\partial(x, y, z)}{\partial(r, v, u)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial u} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial u} \\[2ex] \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial u} \end{vmatrix}.$$

Here

$$\frac{\partial x}{\partial r} = \cos v \cos u, \qquad \frac{\partial x}{\partial v} = -r \sin v \cos u, \qquad \frac{\partial x}{\partial u} = -r\cos v \sin u, \text{ etc.}$$

Taking all nine partial derivatives and simplifying the determinant by applying (5), p. A64, three times gives $J = r^2 \cos v$ (try it!), so that volume element for the sphere is $dV = r^2 \cos v\, dr\, du\, dv$. Here we use the three-dimensional analog to (6), p. 429. Since the radius r is always nonnegative, so that J is nonnegative, we don't need the absolute value of J. This ties in with the discussion on p. 430 following formula (7). Thus the desired triple integral is solved by stepwise integration (verify it!):

$$\int_0^2 \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} r^2 \cos v \, du \, dv \, dr = \int_0^2 \int_{-\pi/2}^{\pi/2} r^2 \cos v \cdot 2\pi \, dv \, dr = \int_0^2 2\pi r^2 \cdot 2 \, dr = 4\pi \cdot \frac{2^3}{3}.$$

(b) Here we use (3*) on p. 443 in Sec. 10.6:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_S \mathbf{F} \cdot \mathbf{N} \, du \, dv = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} (56 \cos^3 v \cos^2 u - 8 \cos v \sin^3 u) \, du \, dv,$$

which, when evaluated, gives the same answer as in (a).

Problem Set 10.7. Page 457

5. Mass distribution. We have to integrate $\sigma = \sin 2x \ \cos 2y$. The given inequalities require that we integrate over y from $\frac{1}{4}\pi - x$ to $\frac{1}{4}\pi$, then over x from 0 to $\frac{1}{4}\pi$, and finally over z from 0 to 6. You should make a sketch to see that the first two ranges of integration form a triangle in the xy-plane with vertices $(\frac{1}{4}\pi, 0)$, $(\frac{1}{4}\pi, \frac{1}{4}\pi)$, and $(0, \frac{1}{4}\pi)$. We need the solution to (A):

$$\text{(A)} \qquad \int_0^6 \int_0^{\pi/4} \int_{(\pi/4)-x}^{\pi/4} \sin 2x \cos 2y \, dy \, dx \, dz.$$

To obtain (A), we first solve the integral over y:

$$\begin{aligned} \text{(B)} \qquad \int_{(\pi/4)-x}^{\pi/4} \sin 2x \cos 2y \, dy &= \sin 2x \int_{(\pi/4)-x}^{\pi/4} \cos 2y \, dy \\ &= \sin 2x \cdot \left[\frac{1}{2} \sin 2y\right]_{(\pi/4)-x}^{\pi/4} \\ &= \sin 2x \cdot \frac{1}{2} \cdot \left[1 - \sin\left(\frac{\pi}{2} - 2x\right)\right] \\ &= \sin 2x \cdot \frac{1}{2} \cdot (1 - \cos 2x) \\ &= \frac{1}{2} \sin 2x (1 - \cos 2x). \end{aligned}$$

Note that the fourth equality in (B) used formula (6), p. A64 in App. A, that is,

$$\sin\left(\frac{\pi}{2} - 2x\right) = \sin \frac{\pi}{2} \cos 2x - \cos \frac{\pi}{2} \sin 2x = 1 \cdot \cos 2x - 0 \cdot \sin 2x = \cos 2x.$$

Next we consider the integral

$$\text{(C)} \qquad \int_0^{\pi/4} \frac{1}{2} \sin 2x \, (1 - \cos 2x) \, dx = \frac{1}{2} \int_0^{\pi/4} \sin 2x \, dx - \frac{1}{2} \int_0^{\pi/4} \sin 2x \cos 2x \, dx.$$

For the second integral (in indefinite form) on the right-hand side of (C),

$$\int \sin 2x \cos 2x \, dx$$

we set $w = \sin 2x$. Then $dw = 2\cos x\, dx$ and the integral becomes

$$\int \sin 2x \cos 2x\, dx = \int \frac{1}{2}\, w\, dw = \frac{1}{2}\frac{w^2}{2} = \frac{1}{4}(\sin 2x)^2 + \text{const.}$$

Hence, continuing our calculation with (C),

$$\begin{aligned}\frac{1}{2}\int_0^{\pi/4} \sin 2x\, dx - \frac{1}{2}\int_0^{\pi/4} \sin 2x \cos 2x\, dx &= -\frac{1}{4}\big[\cos 2x\big]_0^{\pi/4} - \frac{1}{8}\big[(\sin 2x)^2\big]_0^{\pi/4} \\ &= -\frac{1}{4}(0-1) - \frac{1}{8}(1-0) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}.\end{aligned}$$

Finally we solve (D), which gives us the solution to (A):

$$\text{(D)} \qquad \int_0^6 \frac{1}{8}\, dz = \frac{1}{8}\, z\Big|_0^6 = \frac{3}{4}.$$

17. Divergence theorem. We are given $\mathbf{F} = [x^2, \quad y^2, \quad z^2]$ and S the surface of the cone $x^2 + y^2 \le z^2$ where $0 \le z \le h$.

Step 1. Compute div $\mathbf{F}$:

$$\operatorname{div} \mathbf{F} = \operatorname{div}\, [x^2, \quad y^2, \quad z^2] = \frac{\partial F_1}{\partial x} + \frac{\partial F2}{\partial y} + \frac{\partial F3}{\partial z} = 2x + 2y + 2z.$$

Step 2. Find a parametric representation $\mathbf{p}$ for S. Inspired by Example 3, p. 441, a parametric representation of a cone is $x = r\cos v$, $y = r\sin v$, $z = u$, so that

$$\mathbf{p}(r, u, v) = [r\cos v, \quad r\sin v, \quad u].$$

Step 3. Apply the divergence theorem:

$$\text{(E)} \qquad \iint_S \mathbf{F}\bullet\mathbf{n}\, d\mathrm{A} = \iiint_T \operatorname{div} \mathbf{F}\, dV = \iiint_T (2x + 2y + 2z)\, dV.$$

The volume element is

$$dV = r\, dr\, du\, dv$$

so that (E) becomes

$$\iiint_T (2x + 2y + 2z)\, r\, dr\, du\, dv.$$

Remark. We got the factor r into the volume element because of the Jacobian [see (7), p. 430]:

$$\begin{aligned} J &= \frac{\partial(x, y, z)}{\partial(r, v, u)} \\ &= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial u} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial u} \end{vmatrix} \end{aligned}$$

$$= \begin{vmatrix} \cos v & -r\sin v & 0 \\ \sin v & r\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} \cos v & -r\sin v \\ \sin v & r\cos v \end{vmatrix}$$
$$= r\cos^2 v + r\sin^2 v = r.$$

Note that we developed the 3×3 determinant by the last row.

Back to our main computation:

$$\iiint_T (2x + 2y + 2z)\, r\, dr\, du\, dv = \int_{v=0}^{2\pi}\int_{u=0}^{h}\int_{r=0}^{u} (2r\cos v + 2r\sin v + 2u)\, r\, dr\, du\, dv.$$

Note we obtained the limits of integration from the parametrization of **p**. In particular, $r^2 = x^2 + y^2 \le z^2 = u^2$, so that, $r \le z = u$, hence $r \le u$. Furthermore, since $0 \le z \le h$ and $z = u$ we get $0 \le r$. Hence $0 \le r \le u$. The limits of integration for v are just like in a circle $0 \le z \le 2\pi$ (circular! cone). The limits of integration for u are from $u = z$ and $0 \le z \le h$ as given in the problem statement.

After multiplication by r, our integrand will be

$$(2r\cos v + 2r\sin v + 2u)\, r = 2r^2\cos v + 2r^2\sin v + 2ur.$$

First we consider

$$\begin{aligned}\int_{r=0}^{u} (2r^2\cos v + 2r^2\sin v + 2ur)\, dr &= 2\cos v \int_{r=0}^{u} r^2\, dr + 2\sin v \int_{r=0}^{u} r^2\, dr + 2u\int_{r=0}^{u} r\, dr \\ &= 2\cos v\left[\frac{r^3}{3}\right]_0^u + 2\sin v\left[\frac{r^3}{3}\right]_0^u + 2u\left[\frac{r^2}{2}\right]_0^u \\ &= \frac{2}{3}u^3\cos v + \frac{2}{3}u^3\sin v + u^3 \\ &= \frac{2u^3}{3}\left(\cos v + \sin v + \frac{3}{2}\right).\end{aligned}$$

The next integral to be evaluated is

$$\begin{aligned}\int_{u=0}^{h} \frac{2u^3}{3}\left(\cos v + \sin v + \frac{3}{2}\right) du &= \frac{2}{3}\left(\cos v + \sin v + \frac{3}{2}\right)\left[\frac{u^4}{4}\right]_0^h \\ &= \frac{h^4}{6}\left(\cos v + \sin v + \frac{3}{2}\right)\end{aligned}$$

Finally, we obtain the final result for (E), as on p. A26 of the text book:

$$\begin{aligned}\int_{v=0}^{2\pi} \frac{h^4}{6}\left(\cos v + \sin v + \frac{3}{2}\right) dv &= \frac{h^4}{6}\left[\sin v - \cos v + \frac{3}{2}v\right]_0^{2\pi} \\ &= \frac{h^4}{6}\left[\sin 2\pi - \sin 0 - \cos 2\pi + \cos 0 + \frac{3\cdot 2\pi}{2}\right] \\ &= \frac{\pi}{2}\cdot h^4.\end{aligned}$$

21. Moment of inertia. This is an application of triple integrals. The formula for the moment of inertia I_x about the x-axis is given for Probs. 19–23 on p. 458. The region of integration suggests the use of polar coordinates for y and z, that is, cylindrical coordinates with the x-axis as the axis of the cylinder. We set $y = u \cos v$, $z = u \sin v$ and integrate over v from 0 to 2π, over u from 0 to a, and over x from 0 to h. That is, since $y^2 + z^2 = u^2$, we evaluate the triple integral (with u from the element of area $u\,du\,dv$):

$$I_x = \int_{x=0}^{h} \int_{u=0}^{a} \int_{v=0}^{2\pi} u^2 u \, dv \, du \, dx = \int_{x=0}^{h} \int_{u=0}^{a} \int_{v=0}^{2\pi} u^3 \, dv \, du \, dx$$
$$= \int_{x=0}^{h} \int_{u=0}^{a} 2\pi u^3 \, du \, dx = 2\pi \frac{a^4}{4} h.$$

Sec. 10.8 Further Applications of the Divergence Theorem

This section gives three major applications of the divergence theorem:

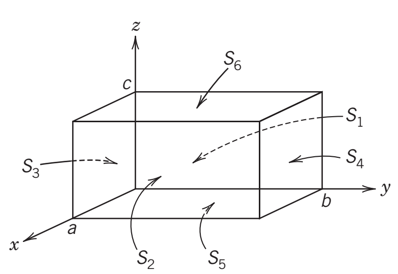
1. **Fluid flow.** Equation (1), p. 458, shows that the divergence theorem gives the flow balance (outflow minus inflow) in terms of an integral of the divergence over the region considered.

2. **Heat equation.** An important application of the divergence theorem is the derivation of the heat equation or diffusion equation (5), p. 460, which we shall solve in Chap. 12 for several standard physical situations.

3. **Harmonic functions.** By the divergence theorem, we obtain basic general properties of harmonic functions (solutions of the Laplace equation with continuous second partial derivatives), culminating in Theorem 3, p. 462.

Problem Set 10.8. Page 462

1. Harmonic functions. Verification of divergence theorem in potential theory. Formula (7), p. 460, is ***another*** *version of the divergence theorem* (Theorem 1, p. 453 of Sec. 10.7) that arises in *potential theory*—the theory of solutions to Laplace's equation (6), p. 460. Note that (7) expresses a very remarkable property of harmonic functions as stated in Theorem 1, p. 460. Of course, (7) can also be used for other functions. The point of the problem is to gain confidence in the formula and to see how to organize more involved calculations so that errors are avoided as much as possible. The box has six faces $S_1, \dots, S_6$, as shown in the figure below.



Sec. 10.8 Prob. 1. Surface S with six faces $S_1, \dots, S_6$

To each of these faces we need normal vectors pointing outward from the faces, that is, away from the box. Looking at the figure above, we see that a normal vector to S_1 pointing into the negative x-direction is $\mathbf{n}_1 = [-1, \quad 0, \quad 0]$. Similarly, a normal vector to S_2 pointing into the positive x-direction is $\mathbf{n}_2 = [1, \quad 0, \quad 0]$, etc. Indeed, all the normal vectors point into negative or positive directions of the axes.

Let us take a careful look at (7), p. 460:

$$\iiint_T \nabla^2 f \, dV = \iint_S \frac{\partial f}{\partial n} \, dA. \tag{7}$$

On the left-hand side of (7) we have that the Laplacian for the given $f = 2z^2 - x^2 - y^2$ is

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -2 + (-2) + 4 = 0 \tag{A}$$

so that the triple integral of (7) is 0.

For the right-hand side of (7) we need the normal derivative of f (see pp. 437 and 460):

$$\frac{\partial f}{\partial n} = (\operatorname{grad} f) \bullet \mathbf{n} = \left[\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}\right] \bullet \mathbf{n}.$$

The gradient of f [see (1), p. 396 of Sec. 9.7]:

$$\operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}\right] = [-2x, \quad -2y, \quad 4z].$$

In the following table we have

$S_1 : x = 0$	$\mathbf{n}_1 \bullet (\operatorname{grad} f) = [-1, \quad 0, \quad 0] \bullet [-2x, \quad -2y, \quad 4z]$ $= 2x = 0 \quad \text{at} \quad x = 0$	$\iint_{S_1} \frac{\partial f}{\partial n} \, dA = 0$
$S_2 : x = a$	$\mathbf{n}_2 \bullet (\operatorname{grad} f) = [1, \quad 0, \quad 0] \bullet [-2x, \quad -2y, \quad 4z]$ $= -2x = -2a \quad \text{at} \quad x = a$	$\iint_{S_2} \frac{\partial f}{\partial n} \, dA = \int_0^c \int_0^b (-2a) \, dy \, dz$ $= \int_0^c (-2ab) \, dz$ $= -2abc$
$S_3 : y = 0$	$\mathbf{n}_3 \bullet (\operatorname{grad} f) = [0, \quad -1, \quad 0] \bullet [-2x, \quad -2y, \quad 4z]$ $= 2y = 0 \quad \text{at} \quad y = 0$	$\iint_{S_3} \frac{\partial f}{\partial n} \, dA = 0$
$S_4 : y = b$	$\mathbf{n}_4 \bullet (\operatorname{grad} f) = [0, \quad 1, \quad 0] \bullet [-2x, \quad -2y, \quad 4z]$ $= -2y = -2b \quad \text{at} \quad y = b$	$\iint_{S_4} \frac{\partial f}{\partial n} \, dA = \int_0^c \int_0^a (-2b) \, dx \, dz$ $= -2bac$
$S_5 : z = 0$	$\mathbf{n}_5 \bullet (\operatorname{grad} f) = [0, \quad 0, \quad -1] \bullet [-2x, \quad -2y, \quad 4z]$ $= 4x = 0 \quad \text{at} \quad z = 0$	$\iint_{S_5} \frac{\partial f}{\partial n} \, dA = 0$
$S_6 : z = c$	$\mathbf{n}_6 \bullet (\operatorname{grad} f) = [0, \quad 0, \quad 1] \bullet [-2x, \quad -2y, \quad 4z]$ $= 4z = 4c \quad \text{at} \quad z = c$	$\iint_{S_6} \frac{\partial f}{\partial n} \, dA = \int_0^b \int_0^a (4c) \, dx \, dy$ $= 4cab$

From the third column of the table we get

$$\begin{aligned}\iint_S \frac{\partial f}{\partial n}\, dA &= \iint_{S_1} \frac{\partial f}{\partial n}\, dA + \cdots + \iint_{S_6} \frac{\partial f}{\partial n}\, dA = 0 + (-2abc) + 0 + (-2bac) + 0 + 4cab \\ &= 0 \\ &= \iiint_T \nabla^2 f\, dV.\end{aligned}$$

We have shown that the six integrals over these six faces add up to zero.

Furthermore, from above, the triple integral is zero since the Laplacian of f is zero as shown in (A); hence f is harmonic. Together, we have established (7) for our special case.

Sec. 10.9 Stokes's Theorem

Stokes's theorem (Theorem 1, p. 464) transforms surface integrals, of a surface S, into line integrals over the boundary curve C of the surface S and, conversely, line integrals into surface integrals. This theorem is a generalization of Green's theorem in the plane (p. 433), as shown in **Example 2**, pp. 466–467. The last part of this section closes a gap we had to leave in (b) of the proof of Theorem 3 (path independence) on p. 423 in Sec. 10.2. Take another look at Sec. 10.2 in the light of Stokes's theorem.

Problem 3 evaluates a surface integral directly and then by Stokes's theorem.

Study hints for Chap. 10. We have reached the end of Chap. 10. Chapter 10 contains a substantial amount of material. For studying purposes, you may want to construct a small table that summarizes line integrals (3), p. 435, Green's theorem in the plane (1), p. 433, divergence theorem of Gauss (2), p. 453, and Stokes's theorem (2), p. 464. This should aid you in remembering the different integrals (line, double, surface, triple) and which theorem to use for transforming integrals. Also look at the **chapter summary** on pp. 470–471 of the textbook.

Problem Set 10.9. Page 468

3. Direct evaluation of a surface integral directly and by using Stokes's theorem. We learned that Stokes's theorem converts surface integrals into line integrals over the boundary of the (portion of the) surface and conversely. It will depend on the particular problem which of the two integrals is simpler, the surface integral or the line integral. (Take a look at solution (a) and (b). Which one is simpler?)

(a) Method 1. Direct evaluation. We are given $\mathbf{F} = [e^{-z}, \quad e^{-z}\cos y, \quad e^{-z}\sin y]$ and the surface $S : z = y^2/2$, with x varying from -1 to 1 and y from 0 to 1. Note that S is a *parabolic cylinder.* Using (1), p. 406, we calculate the curl of $\mathbf{F}$:

$$\text{(A)} \qquad \begin{aligned}\operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{-z} & e^{-z}\cos y & e^{-z}\sin y \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y}(e^{-z}\sin y) - \frac{\partial}{\partial z}(e^{-z}\cos y)\right]\mathbf{i} \\ &\quad - \left[\frac{\partial}{\partial x}(e^{-z}\sin y) - \frac{\partial}{\partial z}e^{-z}\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(e^{-z}\cos y) - \frac{\partial}{\partial y}(e^{-z})\right]\mathbf{k}\end{aligned}$$

$$= [e^{-z}\cos y - (-e^{-z}\cos y)]\mathbf{i} - [0 - (-e^{-z})]\mathbf{j} + (0-0)\mathbf{k}$$
$$= [2e^{-z}\cos y, \quad -e^{-z}, \quad 0].$$

Substituting $z = y^2/2$ into the last line of (A), we obtain the curl $\mathbf{F}$ on S:

$$\text{curl } \mathbf{F} = [2e^{-y^2/2}\cos y, \quad -e^{-y^2/2}, \quad 0].$$

To get a normal vector of S, we write S in the form

$$S: \mathbf{r} = \left[x, \quad y, \quad \tfrac{1}{2}y^2\right].$$

We could also set $x = u$, $y = v$ and write $\mathbf{r} = \left[u, \quad v, \quad \tfrac{1}{2}v^2\right]$, but this would not make any difference in what follows. The partial derivatives are

$$\mathbf{r}_x = [1, \quad 0, \quad 0],$$
$$\mathbf{r}_y = [0, \quad 1, \quad y].$$

We obtain the desired normal vector by taking the cross product of $\mathbf{r}_x$ and $\mathbf{r}_y$:

$$\mathbf{N} = \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & y \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & y \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & y \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$
$$= [0, \quad -y, \quad 1].$$

Setting up the integrand for the surface integral we have

$$(\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dA = (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dx\, dy$$
$$= [2e^{-y^2/2}\cos y, \quad -e^{-y^2/2}, \quad 0] \cdot [0, \quad -y, \quad 1]\, dx\, dy = ye^{-y^2/2}\, dx\, dy.$$

Since x varies between -1 and 1, y between 0 and 1, we obtain the limits of the integrals. We have

$$\text{(B)} \qquad \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dA = \int_{y=0}^{1} \int_{x=-1}^{1} ye^{-y^2/2}\, dx\, dy.$$

Now

$$\int_{x=-1}^{1} ye^{-y^2/2}\, dx = ye^{-y^2/2} \int_{x=-1}^{1} dx = 2ye^{-y^2/2}.$$

Consider

$$\int 2ye^{-y^2/2}\, dy = 2\int ye^{-y^2/2}\, dy = 2\int e^{-w}\, dw = -2e^{-w} = -2e^{-y^2/2},$$

with $w = y^2/2$ so that $dw = y\, dy$. The next step gives us the final answer to (B):

$$\int_{y=0}^{1} 2e^{-y^2/2}\, dy = \left[-2ye^{-y^2/2}\right]_0^1 = 2 - 2e^{-1/2} = 2 - \frac{2}{\sqrt{e}}.$$

The answer on p. A27 also has a second answer with a minus sign, that is, $-(2 - 2/\sqrt{e})$. This answer is obtained if we reverse the direction of the normal vector, that is, we take

$$\widetilde{\mathbf{N}} = -\mathbf{N} = -[0, \quad -y, \quad 1] = [0, \quad y, \quad 1]$$

and proceed as before. (Note that only (a) is required for solving this problem. Same for Probs. 1–10. However, do take a careful look at (b) or, even better, try to solve it yourself and compare. We give (b) for the purpose of learning.)

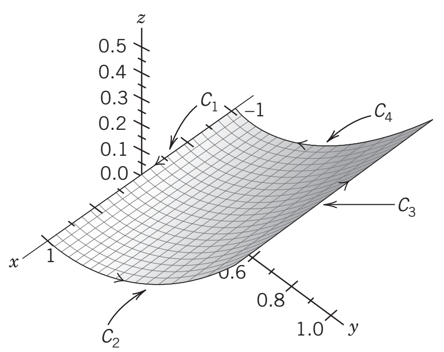
(b) Method 2. By Stokes's Theorem (Theorem 1, p. 464). The theorem gives us

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, dA = \oint_C \mathbf{F} \cdot \mathbf{r}'(s) \, ds. \tag{2}$$

The boundary curve of S has four parts. Sketch the surface S so that you see what is going on. The surface looks like the blade of a snow shovel (see figure below), of the type used in northern and northeastern parts of the United States and in Canada.

The first part, say C_1, is the segment of the x-axis from -1 to 1. On it, x varies from -1 to 1, $y = 0$, and $z = \frac{1}{2}y^2 = 0$. Hence $\mathbf{F}$, evaluated at that segment, is

$$\mathbf{F} = [e^{-z}, \quad e^{-z}\cos y, \quad e^{-z}\sin y] = [1, \quad 1, \quad 0].$$



Sec. 10.9 Prob. 3. Surface S of a cylinder and boundary curves C_1, C_2, C_3, and C_4

Since $s = x$ on C_1, we have the parametrization for C_1:

$$C_1 : \mathbf{r} = [x, \quad 0, \quad 0].$$

and

$$\mathbf{r}' = [1, \quad 0, \quad 0].$$

Hence

$$\mathbf{F} \cdot \mathbf{r}' = [1, \quad 1, \quad 0] \cdot [1, \quad 0, \quad 0] = 1.$$

Then

$$\int_{C_1} \mathbf{F} \cdot \mathbf{r}'(s) \, ds = \int_{x=-1}^{1} dx = 1 - (-1) = 2.$$

The next part of the boundary curve that we will consider, call it C_3, is the upper straight-line edge from $(-1, 1, \frac{1}{2})$ to $(1, 1, \frac{1}{2})$. (This would be the end of the blade of the "snow shovel.") On it, $y = 1$, $z = \frac{1}{2}$, and x varies from 1 to -1 (direction!). Thus $\mathbf{F}$, evaluated at C_3, is

$$\mathbf{F} = [e^{-z}, \quad e^{-z}\cos y, \quad e^{-z}\sin y] = [e^{-1/2}, \quad e^{-1/2}\cos 1, \quad e^{-1/2}\sin 1].$$

We can represent C_3 by $\mathbf{r} = [x, \quad 1, \quad \frac{1}{2}]$ (since $y = 1$, $z = \frac{1}{2}$, with x varying as determined before). Also $\mathbf{r}' = [1, \quad 0, \quad 0]$. We are ready to set up the integrand:

$$\mathbf{F} \bullet \mathbf{r}' = [e^{-1/2}, \quad e^{-1/2}\cos 1, \quad e^{-1/2}\sin 1] \bullet [1, \quad 0, \quad 0] = e^{-1/2}.$$

Then

$$\int_{C_3} \mathbf{F} \bullet \mathbf{r}'(s)\, ds = \int_{x=1}^{-1} e^{-1/2} dx = e^{-1/2} \int_{x=1}^{-1} dx = (-1-1)e^{-1/2} = -2e^{-1/2}.$$

The sum of the two line integrals over C_1 and C_3, respectively, equals $2 - 2e^{-1/2}$, which is the result as before.

But we are not yet finished. We have to show that the sum of the other two integrals over portions of parabolas is zero.

We now consider the curved parts of the surface. First, C_2 (one of the sides of the blade of the "snow shovel") is the parabola $z = \frac{1}{2}y^2$ in the plane $x = 1$, which we can represent by

$$C_2 : \mathbf{r} = \left[1, \quad y, \quad \tfrac{1}{2}y^2\right].$$

The derivative with respect to y is

$$\mathbf{r}' = [0, \quad 1, \quad y].$$

Furthermore, $\mathbf{F}$ on C_2 is

$$\mathbf{F} = [e^{-(1/2)y^2}, \quad e^{-(1/2)y^2}\cos y, \quad e^{-(1/2)y^2}\sin y]$$

and

$$\begin{aligned}\mathbf{F} \bullet \mathbf{r}' &= [e^{-(1/2)y^2}, \quad e^{-(1/2)y^2}\cos y, \quad e^{-(1/2)y^2}\sin y] \bullet [0, \quad 1, \quad y] \\ &= e^{-(1/2)y^2}\cos y + y\, e^{-1/2y^2}\sin y.\end{aligned}$$

This must be integrated over y from 0 to 1, that is,

$$\int_{C_2} \mathbf{F} \bullet \mathbf{r}'(s)\, ds = \int_{y=0}^{1} (e^{-(1/2)y^2}\cos y + y\, e^{-(1/2)y^2}\sin y)\, dy.$$

However, for the fourth portion C_4, we obtain exactly the same expression because C_4 can be represented by

$$C_4 : \mathbf{r} = \left[-1, \quad y, \quad \tfrac{1}{2}y^2\right] \qquad \text{so that} \qquad \mathbf{r}' = [0, \quad 1, \quad 2y].$$

Furthermore, here $\mathbf{F} \cdot \mathbf{r}'$ is the same as for C_2, because $\mathbf{F}$ does not include x. Now, on C_4, we have to integrate over y in the opposite sense from 1 to 0, so that the two integrals do indeed cancel each other and their sum is zero.

Conclusion. We see, from this problem, that the evaluation of the line integral may be more complicated if the boundary is more complicated. On the other hand, if the curl is complicated or if no convenient parametric representation of the surface integral can be found, then the evaluation of the line integral may be simpler.

Notice that for (b), Stokes's theorem, we integrate along the boundary curves C_1, C_2, C_3, and C_4 in the directions marked with arrows in the figure.

13. Evaluation of a line integral using Stoke's theorem. First approach by using formula (2), p. 464. We convert the given line integral into a surface integral by Stokes's theorem (2):

$$\text{(2)} \qquad \oint_C \mathbf{F} \cdot \mathbf{r}'(s)\, ds = \iint_S (\text{curl}\, \mathbf{F}) \cdot \mathbf{n}\, d\text{A}.$$

We define the surface S bounded by $C : x^2 + y^2 = 16$ to be the circular disk $x^2 + y^2 \leq 16$ in the plane $z = 4$. We need the right-hand side of (2) with given $\mathbf{F} = [-5y, \quad 4x, \quad z]$:

$$\text{(A)} \qquad \text{curl}\, \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -5y & 4x & z \end{vmatrix} \qquad \text{[by (1), p. 406]}$$

$$= \left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}4x\right)\mathbf{i} - \left[\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}(-5y)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}4x - \frac{\partial}{\partial y}(-5y)\right]\mathbf{k}$$

$$= (0-0)\mathbf{i} + (0-0)\mathbf{j} + [4-(-5)]\mathbf{k}$$

$$= [0, \quad 0, \quad 9] \qquad \text{on } S.$$

From the assumption of right-handed Cartesian coordinates and the z-component of the surface normal to be nonnegative, we have, for this problem, a unit normal vector $\mathbf{n}$ to the given circular disk in the given plane:

$$\text{(B)} \qquad \mathbf{n} = \mathbf{k} = [0, \quad 0, \quad 1].$$

Hence the dot product [formula (2), p. 361]:

$$\text{(C)} \qquad (\text{curl}\, \mathbf{F}) \cdot \mathbf{n} = [0, \quad 0, \quad 9] \cdot [0, \quad 0, \quad 1] = 9.$$

Hence the right-hand side of (2) is

$$\iint_S (\text{curl}\, \mathbf{F}) \cdot \mathbf{n}\, dA = \iint_S 9\, dA$$

$$= 9 \cdot [\text{Area of the given circular disk } x^2 + y^2 \leq 4^2]$$

$$= 9 \cdot \pi r^2 \qquad \text{with radius } r = 4$$

$$= 9 \cdot \pi \cdot 4^2$$

$$= 144\pi.$$

Remark. Note that the calculations (A), (B), and (C) can be simplified, if we first determine that $\mathbf{n} = \mathbf{k}$, so that then $(\text{curl } \mathbf{F}) \cdot \mathbf{n}$ is simply the component of curl $\mathbf{F}$ in the positive z-direction. Since $\mathbf{F}$, with $z = 4$, has the components $\mathbf{F}_1 = -5y$, $\mathbf{F}_2 = 4x$, $\mathbf{F}_3 = 4$ we obtain the same value as before:

$$(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 4 - (-5) = 9.$$

Second approach by using formula (2*), p. 464. In this approach, we can use the plane of the circle given by $\mathbf{r} = [x(u, v),\quad y(u, v),\quad z(u, v)] = [u,\quad v,\quad 4]$. Then $\mathbf{r}_u = [1,\quad 0,\quad 0]$ and $\mathbf{r}_v = [0,\quad 1,\quad 0]$. This gives us a normal vector $\mathbf{N}$ to the plane:

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} = \mathbf{k} = [0,\quad 0,\quad 1].$$

Hence

$$(\text{curl } \mathbf{F}) \cdot \mathbf{N} = [0,\quad 0,\quad 9] \cdot [0,\quad 0,\quad 1] = 9.$$

Now 9 times the area of the region of integration in the uv-plane, which is the interior of the circle $u^2 + v^2 = 16$ and has area $\pi \cdot (\text{radius})^2 = \pi \cdot 4^2 = 16\pi$, so that the answer is $16\pi \cdot 9 = 144\pi$, as before.

Solution for the Double Integral Problem (see p. 176 of the Student Solutions Manual). The shaded area in Fig. 10.3 (a) and (b) is the area of integration. We note that, for $0 < x < 1$, the straight line is above the parabola.

There are two ways to solve the problem. Perhaps the easier way is to integrate *first* in the y-direction (vertical direction) from $y = x^2$ to $y = x$ and *then* in the x-direction (horizontal direction) from $x = 0$ to $x = 1$. See Fig. 10.3(a). [Conceptually, this is (3), p. 427, and Fig. 229, p. 428]:

$$\begin{aligned} \int_0^1 \int_{x^2}^{x} x^3\, dy\, dx &= \int_{x=0}^{1} x^3 \left(\int_{y=x^2}^{x} dy \right) dx \\ &= \int_{x=0}^{1} x^3 [y]_{y=x^2}^{y=x}\, dx \\ &= \int_0^1 [x^3(x - x^2)]\, dx \\ &= \int_0^1 x^4\, dx - \int_0^1 x^5\, dx \\ &= \left[\frac{x^5}{5}\right]_0^1 - \left[\frac{x^6}{6}\right]_0^1 \\ &= \frac{1}{5} - \frac{1}{6} = \frac{6}{30} - \frac{5}{30} \\ &= \frac{1}{30} = 0.03333. \end{aligned}$$

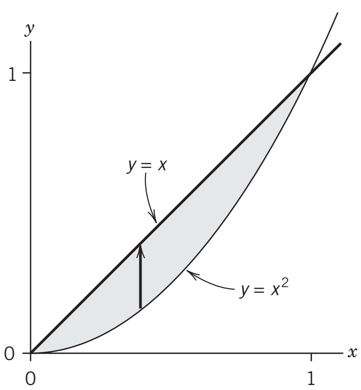


Fig. 10.3(a). Integrating first in y-direction and then in x-direction

We can also integrate in the reverse order: first integrating in the x-direction (horizontal direction) from $x = y$ to $x = \sqrt{y}$ and then integrating in the y-direction from 0 to 1. Because we are integrating first over x, the limits of integration must be expressed in terms of y. Thus $y = x^2$ becomes $x = \sqrt{y}$ (see (4) and Fig. 230, p. 428, for conceptual understanding):

$$\int_0^1 \int_y^{\sqrt{y}} x^3\,dx\,dy = \int_{y=0}^1 \left(\int_{x=y}^{\sqrt{y}} x^3\,dx\right) dy$$

$$= \int_{y=0}^1 \left[\frac{x^4}{4}\right]_{x=y}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{y^2}{4} - \frac{y^4}{4}\right) dy$$

$$= \frac{1}{4}\left(\int_0^1 y^2\,dy - \int_0^1 y^4\,dy\right)$$

$$= \frac{1}{4}\left(\left[\frac{y^3}{3}\right]_0^1 - \left[\frac{y^5}{5}\right]_0^1\right)$$

$$= \frac{1}{4}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{1}{4}\left(\frac{5}{15} - \frac{3}{15}\right)$$

$$= \frac{1}{4} \cdot \frac{2}{15} = \frac{2}{60}$$

$$= \frac{1}{30} = 0.03333.$$

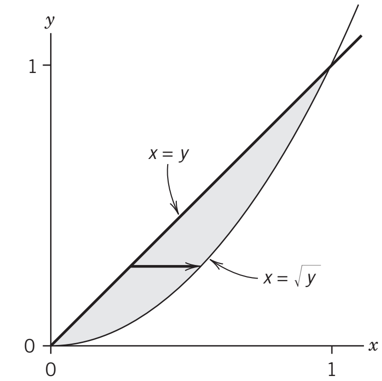


Fig. 10.3(b). Integrating first in x-direction and then in y-direction

PART C

Fourier Analysis. Partial Differential Equations (PDEs)

Chap. 11 Fourier Analysis

Rotating parts of machines, alternating electric circuits, and motion of planets are just a few of the many periodic phenomena that appear frequently in engineering and physics. The physicist and mathematician Jean-Baptiste Joseph Fourier (see Footnote 1, p. 473 of textbook) had the brilliant idea that, in order to model such problems effectively, **difficult periodic functions could be represented by *simpler* periodic functions that are combinations of *sine* and *cosine* functions.** These representations are infinite series called **Fourier series**. Fourier's insight revolutionized applied mathematics and bore rich fruits in many areas of mathematics, engineering, physics, and other fields. Examples are vibrating systems under forced oscillations, electric circuits (Sec. 11.3), Sturm–Liouville problems, vibrating strings (11.5), convolution, discrete and fast Fourier transforms (Sec. 11.9). Indeed, Fourier analysis is such an important field that we devote a whole chapter to it and also use it in Chap. 12.

Section 11.1, pp. 474–483 on Fourier series and orthogonality is the foundation from which the rest of the chapter evolves. Sections 11.2–11.4 complete the discussion of Fourier series. The concept of orthogonality of systems of functions leads to the second topic of *Sturm–Liouville expansions* in Secs. 11.5 and 11.6, pp. 498–510. In another line of thought, the concept of Fourier series leads directly to *Fourier integrals and transforms* in Secs. 11.7–11.9, pp. 510–533.

In terms of prior knowledge, you should know the details about the sine and cosine functions (if you feel the need, review pp. A63–A64 in Sec. A3.1 of App. 3 of the textbook). *You should have a good understanding of how to do* **integration by parts** *from calculus* (see inside the front cover of the text) *as you will have to use it frequently throughout the chapter*, starting right in the first section. (However, in some of our answers to solved problems, integration by parts is shown in detail.) Some knowledge of nonhomogeneous ODEs (review Secs. 2.7–2.9) for Sec. 11.3 (pp. 492–495) and homogeneous ODEs (review Sec. 2.2) for Secs. 11.5 (pp. 498–504) and 11.6 (pp. 504–510). Some knowledge of special functions (refer back to Secs. 5.2–5.5, as needed) and a modest understanding of eigenvalues for Sec. 11.5 (pp. 498–504) would be useful.

Sec. 11.1 Fourier Series

Important examples of **periodic functions** are $\cos nx$, $\sin nx$, where $n = 1, 2, 3, \ldots$ (natural numbers), which have a period of $2\pi/n$ and hence also a period of 2π. A *trigonometric system* is obtained if we include the constant function 1, as given by (3), p. 475, and shown in Fig. 259. Next we use (3) to build a system of trigonometric series, the famous **Fourier series** (5), p. 476, whose coefficients are determined by the Euler formulas (6). **Example 1**, pp. 477–478, and **Prob. 13** show how to use (6) to calculate Fourier series. Take a careful look. The integration uses substitution and integration by parts and is a little bit different from what you are used to in calculus because of the n. Formula (6.0) gives the constant a_0; here, $a_0 = 0$. Equation (6a) gives the cosine coefficients $a_1, a_2, a_3, a_4, a_5, \ldots$ and (6b) the sine coefficients $b_1, b_2, b_3, b_4, b_5, \ldots$ of the Fourier series. **You have to solve a few Fourier series problems so that (5) and (6) stick to your memory. It is unlikely that in a closed book exam you will be able to derive (5) and (6).** The periodic function in typical problems are given either algebraically (such as Probs. 12–15, p. 482) or in terms of graphs (Probs. 16–21).

The rest of Sec. 11.1 justifies, theoretically, the use of Fourier series. Theorem 1, p. 479, on the **orthogonality** of (3), is used to prove the Euler formulas (6). Theorem 2, p. 480, accounts for the great generality of Fourier series.

Problem Set 11.1. Page 482

3. Linear combinations of periodic functions. Vector space. When we add periodic functions with the same period p, we obtain a periodic function with that period p. Furthermore, when we multiply a function of period p by a constant, we obtain a function that also has period p. Thus all functions of period p form an important example of a *vector space* (see pp. 309–310 of textbook).

More formally, if (i) $f(x)$ and $g(x)$ have period p, then (ii) for any constants a and b, $h(x) = af(x) + bg(x)$ has the period p.

Proof. Assume (i) holds. Then by (2), p. 475, with $n = 1$ we have

$$f(x+p) = f(x), \qquad g(x+p) = g(x). \tag{A}$$

Take any linear combination of these functions, say

$$h = af + bg \quad (a, b \text{ constant}).$$

To show (ii) we must show that

$$h(x+p) = h(x). \tag{B}$$

Now

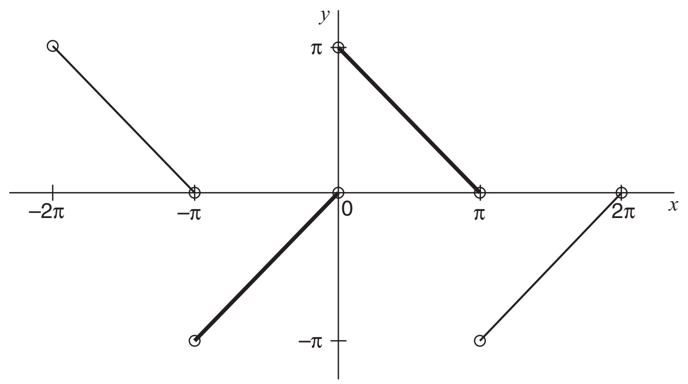
$$\begin{aligned} h(x+p) &= af(x+p) + bg(x+p) && \text{(by definition of } h) \\ &= af(x) + bg(x) && \text{[by (A)]} \\ &= h(x) && \text{(by definition of } h). \end{aligned}$$

Hence (B) holds.

Note that when doing such proofs you have to strictly follow the definitions.

9. Graph of function of period 2π. We want to graph $f(x)$, which is assumed to have a period of 2π:

$$f(x) = \begin{cases} x & -\pi < x < 0, \\ \pi - x & 0 < x < \pi. \end{cases}$$



Sec. 11.1 Prob. 9. Graph of given function $f(x)$ with period 2π. Note the discontinuity at $x = 0$. Since the function is periodic with period 2π, discontinuities occur at $\ldots, -4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$

In the first part of the definition of f as $-\pi < x < 0, f(x) = x$ varies $-\pi < f(x) < 0$. Graphically, the function is a straight line segment sloping upward from (but not including) the point $(-\pi, \pi)$ to (not including) the origin. In the second part as $0 < x < \pi, f(x) = \pi - x$ varies $0 < f(x) < \pi$. Here the function is a straight line segment sloping downward from (but not including) $(0, \pi)$ to (not including) $(\pi, 0)$. There is a jump at $x = 0$. This gives us the preceeding graph with the smallest period (defined on p. 475) in a thick line and the extension outside that period by a thin line. Notice that Fourier series allow *discontinuous* periodic functions.

Note: To avoid confusion, the thick line is the given function f with the given period 2π. So, in your answer, you would not have to show the extension.

13. Determination of Fourier series. We compute in complete detail the Fourier series of function $f(x)$ defined and graphed in **Prob. 9** above.

Step 1. Compute the coefficient a_0 of the Fourier series by (6.0), p. 476.

Using (6.0), we obtain by familiar integration

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi}(\pi - x)\,dx\right)$$

$$= \frac{1}{2\pi}\left(\frac{x^2}{2}\bigg|_{-\pi}^{0} + \left[\pi x - \frac{x^2}{2}\right]_0^{\pi}\right) = \frac{1}{2\pi}\left(-\frac{\pi^2}{2} + \left[\pi^2 - \frac{\pi^2}{2}\right]\right) = \frac{1}{2\pi}\left(-\pi^2 + \pi^2\right) = 0.$$

This answer makes sense as the area under the curve of $f(x)$ between $-\pi$ and π (taken with a minus sign where $f(x)$ is negative) is zero, as clearly seen from the graph in Prob. 9.

Step 2. Compute the cosine coefficients a_n, where $n = 1, 2, 3, \ldots$ (natural numbers) of the Fourier series by (6a), p. 476.

Using (6a), we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\cos nx\,dx + \int_{0}^{\pi}(\pi - x)\cos nx\,dx\right].$$

The integration is a bit different from what you are familiar with in calculus because of the n. We go slowly and evaluate the corresponding indefinite integrals first (ignoring the constants of integration) before evaluating the definite integrals. To evaluate $\int x \cos nx\,dx$ we set

$$u = nx; \qquad \frac{du}{dx} = n; \qquad du = n\,dx;$$
$$x = \frac{u}{n}.$$

Hence

$$\int x \cos nx\,dx = \int (x \cos nx)\frac{1}{n}\,n\,dx = \frac{1}{n}\int \frac{u}{n}\cos u\,du = \frac{1}{n^2}\int u \cos u\,du.$$

Integration by parts yields

$$\int u \cos u\,du = u \sin u - \int 1 \cdot \sin u\,du = u \sin u - 1 \cdot (-\cos u) = u \sin u + \cos u.$$

Remembering what the substitution was,

$$\int x \cos nx\,dx = \frac{1}{n^2}(nx \sin nx + \cos nx) = \frac{1}{n} x \sin nx + \frac{1}{n^2}\cos nx.$$

The corresponding definite integral is

$$\int_{-\pi}^{0} x \cos nx\,dx = \left[\frac{1}{n} x \sin nx + \frac{1}{n^2}\cos nx\right]_{-\pi}^{0}.$$

The upper limit of integration of 0 gives

$$\frac{1}{n^2}\cos 0 = \frac{1}{n^2} \cdot 1 = \frac{1}{n^2}.$$

The lower limit of $-\pi$ gives

$$\left(\frac{1}{n}\right)(-\pi)[\sin(-n\pi)] + \frac{1}{n^2}\cos(-n\pi).$$

This lower limit can be simplified by noticing that

$$\sin(-n\pi) = -\sin(n\pi),$$

and that

$$\sin(n\pi) = 0 \qquad \text{for all } n = 1, 2, 3, \ldots \text{ (natural numbers).}$$

Similarly, for cos

$$\cos(-n\pi) = \cos(n\pi).$$

Thus the lower limit simplifies to

$$\frac{1}{n^2}\cos n\pi.$$

Subtracting the upper limit from the lower limit, we get

$$\int_{-\pi}^{0} x \cos nx \, dx = \frac{1}{n^2} - \frac{1}{n^2} \cos n\pi.$$

The next integral breaks into two integrals:

$$\int_{0}^{\pi} (\pi - x) \cos nx \, dx = \pi \int_{0}^{\pi} \cos nx \, dx - \int_{0}^{\pi} x \cos nx \, dx.$$

Using the substitution $u = nx$ as before, the first of these, in indefinite form, is

$$\int \cos nx \, dx = \int \frac{1}{n} \cos nx \, n \, dx = \frac{1}{n} \int \cos u \, du = \frac{1}{n} \sin u = \frac{1}{n} \sin nx.$$

Hence

$$\int_{0}^{\pi} \cos nx \, dx = \frac{1}{n} \sin nx \bigg|_{0}^{\pi} = \frac{1}{n} \sin n\pi - \frac{1}{n} \sin 0 = 0.$$

Using the calculations from before, we get

$$\begin{aligned} \int_{0}^{\pi} x \cos nx \, dx = \left[\frac{1}{n} x \sin nx + \frac{1}{n^2} \cos nx \right]_{0}^{\pi} &= \frac{1}{n} \pi \sin n\pi + \frac{1}{n^2} \cos n\pi - \frac{1}{n^2} \cos 0 \\ &= \frac{1}{n^2} \cos n\pi - \frac{1}{n^2}. \end{aligned}$$

Putting the three definite integrals together, we have

$$\begin{aligned} a_n &= \frac{1}{\pi} \left(\int_{-\pi}^{0} x \cos nx \, dx + \pi \int_{0}^{\pi} \cos nx \, dx - \int_{0}^{\pi} x \cos nx \, dx \right) \\ &= \frac{1}{\pi} \left[\frac{1}{n^2} - \frac{1}{n^2} \cos n\pi + \pi \cdot 0 - \left(\frac{1}{n^2} \cos n\pi - \frac{1}{n^2} \right) \right] \\ &= \frac{1}{\pi} \left(\frac{1}{n^2} - \frac{1}{n^2} \cos n\pi - \frac{1}{n^2} \cos n\pi + \frac{1}{n^2} \right) \\ &= \frac{1}{\pi} \left(\frac{2}{n^2} - \frac{2}{n^2} \cos n\pi \right) \\ &= \frac{2}{\pi n^2} (1 - \cos n\pi). \end{aligned}$$

We substitute $n = 1, 2, 3, 4, 5 \ldots$ in the previous formula and get the cosine coefficients $a_1, a_2, a_3, a_4, a_5, \ldots$ of the Fourier series, that is, for

$$\begin{aligned} n = 1 \qquad & a_1 = \frac{2}{\pi}(1 - \cos \pi) = \frac{2}{\pi}[1 - (-1)] = \frac{4}{\pi}, \\ n = 2 \qquad & a_2 = \frac{2}{\pi \cdot 2^2}(1 - \cos 2\pi) = \frac{1}{2\pi}(1 - 1) = 0, \end{aligned}$$

$$\begin{array}{ll} n=3 & a_3 = \dfrac{2}{\pi \cdot 3^2}(1-\cos 3\pi) = \dfrac{1}{9\pi}[1-(-1)] = \dfrac{4}{9\pi}, \\ n=4 & a_4 = 0, \\ n=5 & a_5 = \dfrac{2}{25\pi}[1-(-1)] = \dfrac{4}{25\pi}, \\ \cdots & \cdots \end{array}$$

$$\begin{aligned} & a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x + a_5 \cos 5x + \cdots \\ &= \frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x + \frac{4}{25\pi}\cos 5x + \cdots \\ &= \frac{4}{\pi}\Big(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\Big). \end{aligned}$$

Step 3. Compute the sine coefficients b_n, where $n = 1, 2, 3, \ldots$ (natural numbers) of the Fourier series by (6b), p. 476.

Formula (6b) gives us

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\sin nx\,dx + \int_{0}^{\pi}(\pi - x)\sin nx\,dx\right].$$

As before, we set $u = nx$ and get

$$\int x\sin nx\,dx = \int \frac{1}{n}(x\sin nx)n\,dx = \frac{1}{n}\int \frac{u}{n}\sin u\,du = \frac{1}{n^2}\int u\sin u\,du.$$

Integration by parts gives us

$$\int u\sin u\,du = u(-\cos u) - \int 1\cdot(-\cos u)\,du = -u\cos u - (-\sin u) = -u\cos u + \sin u.$$

Expressing it in terms of x gives us

$$\int x\sin nx\,dx = \frac{1}{n^2}(-u\cos u + \sin u) = \frac{1}{n^2}(-nx\cos nx + \sin nx) = -\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx.$$

The corresponding definite integral evaluates to

$$\begin{aligned} \int_{-\pi}^{0} x\sin nx\,dx &= \left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{0} \\ &= -\left[-\frac{1}{n}(-\pi)\cos(-n\pi) + \frac{1}{n^2}\sin(-n\pi)\right] \qquad \text{(upper limit contributes 0)} \\ &= -\frac{\pi}{n}\cos(-n\pi) - \frac{1}{n^2}\sin(-n\pi) \\ &= -\frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\sin n\pi \\ &= -\frac{\pi}{n}\cos n\pi, \end{aligned}$$

where again we used that $\cos(-n\pi) = \cos(n\pi)$, $\sin(-n\pi) = -\sin(n\pi)$, and $\sin(n\pi) = 0$ for natural numbers n.

Next

$$\int_0^{\pi} (\pi - x) \sin nx \, dx = \pi \int_0^{\pi} \sin nx \, dx - \int_0^{\pi} x \sin nx \, dx.$$

$$\int \sin nx \, dx = -\frac{1}{n} \cos nx;$$

$$\int_0^{\pi} \sin nx \, dx = \left[-\frac{1}{n} \cos nx \right]_0^{\pi} = -\frac{1}{n} \cos n\pi - \left(-\frac{1}{n} \cos 0 \right) = -\frac{1}{n} \cos n\pi + \frac{1}{n}.$$

Also, from before,

$$\int_0^{\pi} x \sin nx \, dx = \left[-\frac{1}{n} x \cos nx + \frac{1}{n^2} \sin nx \right]_0^{\pi} = -\frac{1}{n} \pi \cos n\pi + \frac{1}{n^2} \sin n\pi - 0 = -\frac{1}{n} \pi \cos n\pi.$$

Thus

$$\begin{aligned} \int_0^{\pi} (\pi - x) \sin nx \, dx &= \pi \int_0^{\pi} \sin nx \, dx - \int_0^{\pi} x \sin nx \, dx \\ &= \pi \left(-\frac{1}{n} \cos n\pi + \frac{1}{n} \right) - \left(-\frac{1}{n} \pi \cos n\pi \right) \\ &= -\frac{1}{n} \pi \cos n\pi + \frac{\pi}{n} + \frac{1}{n} \pi \cos n\pi \\ &= \frac{\pi}{n}. \end{aligned}$$

We get

$$\begin{aligned} b_n &= \frac{1}{\pi} \left[\int_{-\pi}^{0} x \sin nx \, dx + \int_0^{\pi} (\pi - x) \sin nx \, dx \right] \\ &= \frac{1}{\pi} \left(-\frac{\pi}{n} \cos n\pi + \frac{\pi}{n} \right) \\ &= -\frac{1}{n} \cos n\pi + \frac{1}{n}. \end{aligned}$$

We substitute $n = 1, 2, 3, 4, 5 \ldots$ in the previous formula and get the sine coefficients $b_1, b_2, b_3, b_4, b_5, \ldots$ of the Fourier series, that is, for

$$\begin{array}{ll} n = 1 & b_1 = -\cos \pi + 1 = 1 + 1 = 2, \\ n = 2 & b_2 = -\frac{1}{2} \cos 2\pi + \frac{1}{2} = -\frac{1}{2} \cdot 1 + \frac{1}{2} = 0, \\ n = 3 & b_3 = -\frac{1}{3} \cos 3\pi + \frac{1}{3} = -\frac{1}{3}(-1) + \frac{1}{3} = \frac{2}{3}, \\ n = 4 & b_4 = -\frac{1}{4} \cos 4\pi + \frac{1}{4} = -\frac{1}{4} \cdot 1 + \frac{1}{4} = 0, \\ n = 5 & b_5 = -\frac{1}{5} \cos 5\pi + \frac{1}{5} = -\frac{1}{5}(-1) + \frac{1}{5} = \frac{2}{5}, \\ \ldots & \ldots \end{array}$$

$$\begin{aligned}
&b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + b_5 \sin 5x + \cdots \\
&= 2 \sin x + \tfrac{2}{3} \sin 3x + \tfrac{2}{5} \sin 5x + \cdots \\
&= 2\big(\sin x + \tfrac{1}{3} \sin 3x + \tfrac{1}{5} \sin 5x + \cdots\big).
\end{aligned}$$

Step 4. Using the results from Steps 1–3, which used (*6.0*), (*6a*), *and* (*6b*), *p. 476, write down the final answer, that is, the complete Fourier series of the given function.* Taking it all together, the desired Fourier series of $f(x)$ is

$$\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \\
&= 0 + \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right) + 2\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).
\end{aligned}$$

You should sketch or graph some partial sums, to see how the Fourier series approximates the discontinuous periodic function. See also Fig. 261, p. 478.

Note that $\cos n\pi = -1$ for odd n and $\cos n\pi = 1$ for even n, so we could have written more succinctly

$$\cos n\pi = (-1)^n \qquad \text{where } n = 1, 2, 3, \ldots$$

in some of the formulas above.

Be aware that in **Probs. 16–21** you first have to find a formula for the given function before being able to calculate the Fourier coefficients.

Sec. 11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions

Section 11.2 expands Sec. 11.1 in three straightforward ways. First, we generalize the period of the Fourier series from $p = 2\pi$ to an *arbitrary* period $p = 2L$. This gives (5) and (6), p. 484, which are slightly more difficult than related (5) and (6) of Sec. 11.1. This is illustrated in **Prob. 11** and by the periodic rectangular wave in **Examples 1** and **2**, pp. 484–485, and the half-wave rectifier in **Example 3**.

Second, we discuss, systematically, what you have noticed before, namely, that an even function has a Fourier cosine series (5*), p. 486 (no sine terms). An odd function has a Fourier sine series (5**) (no cosine terms, no constant term).

Third, we take up the situation in many applications, namely, that a function is given over an interval of length L and you can develop it either into a cosine series (5*), p. 486, of period $2L$ (not L) or a sine series (5**) of period $2L$; so here you have freedom to decide one way or another. The resulting Fourier series are called **half-range expansions** and are calculated in **Example 6**, pp. 489–490.

Problem Set 11.2. Page 490

3. Sums and products of even functions. We claim that the *sum of two even functions is even*. The approach we use is to strictly apply definitions and arithmetic. (Use this approach for **Probs. 3–7**).

Proof. Let f, g be any two even functions defined on a domain that is common (shared by) both functions.

Then, by p. 486 (definition of even function),

(E1) $$f(-x) = f(x); \qquad g(-x) = g(x).$$

Let h be the sum of f and g, that is,

(E2) $$h(x) = f(x) + g(x).$$

Consider $h(-x)$:

$$\begin{aligned} h(-x) &= f(-x) + g(-x) && \text{[by (E2)]} \\ &= f(x) + g(x) && \text{[by (E1)]} \\ &= h(x) && \text{[by (E2)].} \end{aligned}$$

Complete the problem for products of even functions.

11. Fourier series with arbitrary period $p = 2L$. Even, odd, or neither? $f(x) = x^2$ $(-1 < x < 1)$, $p = 2$. To see whether f is even or odd or neither we investigate

$$f(-x) = (-x)^2 = (-1)^2 x^2 = x^2 = f(x) \quad \text{on the domain} \quad -1 < x < 1.$$

Thus $f(-x) = f(x)$ which, by definition (see p. 486), means that f is an even function.

Aside: more general remark. The function $g(x) = x^3$ where $-a < x < a$ for any positive real number a is an odd function since

$$g(-x) = (-x)^3 = (-1)^3 x^3 = -x^3 = -g(x).$$

This suggests that any function x^n, where n is a natural number and $-a < x < a$ (as before) is an even function if n is even and an odd function if n is odd.

Caution: The interval on which g is defined is important! For example, $g(x) = x^3$ on $0 < x < a$, as before, is neither odd nor even. (Why?)

Fourier series. $p = 2 = 2L$ means that $L = 1$. Since f is even, we calculate a_0 by (6*), p. 486:

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{1}\int_0^1 x^2\,dx = \frac{x^3}{3}\bigg|_0^1 = \frac{1}{3}.$$

We obtain the cosine coefficients by (6*), p. 486, and evaluate the integral by integration by parts

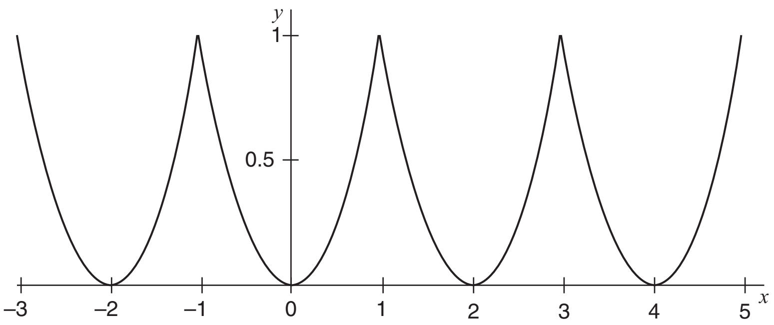
$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = 2\int_0^1 x^2 \cos n\pi x\,dx \\ &= 2\frac{x^2}{n\pi}\sin n\pi x\bigg|_0^1 - \frac{2}{n\pi}\int_0^1 2x\sin n\pi x\,dx. \end{aligned}$$

The first term in the last equality is zero. The integral is calculated by another integration by parts:

$$-\frac{2}{n\pi}\int_0^1 2x\sin n\pi x\,dx = \frac{4}{(n\pi)^2}x\cos n\pi x\bigg|_0^1 - \frac{4}{(n\pi)^2}\int_0^1 \cos n\pi x\,dx.$$

The integral is zero, as you should verify. The lower limit of the first term is 0. Hence we have

$$\frac{4}{(n\pi)^2}x\cos n\pi x\bigg|_0^1 = \frac{4}{n^2\pi^2}\cos n\pi = \frac{4}{n^2\pi^2}(-1)^n.$$



Sec. 11.2 Prob. 11. Given periodic function $f(x) = x^2$ of period $p = 2$

This gives us the desired Fourier series:

$$\frac{1}{3} + \frac{4}{\pi^2}\left(-\cos \pi x + \frac{1}{4}\cos 2\pi x - \frac{1}{9}\cos 3\pi x + - \cdots\right).$$

Remark. Since f was even, we were able to use the simpler formulas of (6*), p. 486, instead of the more general and complicated formulas (6), p. 484. The reason is given by (7a) and the text at the bottom of p. 486. In brief, since f is even, it is represented by a Fourier series of even functions, hence cosines. Furthermore, for an even function f, the integral $\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$.

19. Application of Fourier series. Trigonometric formulas. Our task is to show that

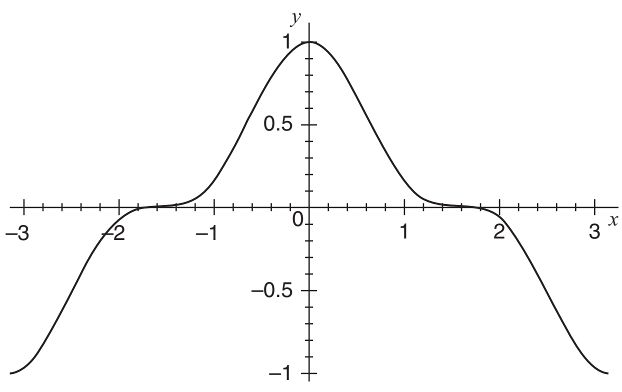
$$\text{(A)} \qquad \cos^3 x = \tfrac{3}{4}\cos x + \tfrac{1}{4}\cos 3x$$

using Fourier series. To obtain the formula for $\cos^3 x$ from the present viewpoint, we calculate $a_0 = 0$, the average of $\cos^3 x$ (see the figure). Next we have

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^3 x\,\cos x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^4 x\,dx = \frac{1}{\pi}\cdot\frac{3\pi}{4} = \frac{3}{4}.$$

In a similar vein, calculations show $a_2 = 0$. We can also show that

$$a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^3 x\,\cos 3x\,dx = \frac{1}{4}.$$



Sec. 11.2 Prob. 19. Graph of $\cos^3 x$

Also, $a_4 = 0$ and so further Fourier coefficients are zero. This leads to (A). Proceed similarly for $\sin^3 x$ and $\cos^4 x$. Compare your answer for $\cos^4 x$ with the one given on p. A28 of App. 2. Here is a rare instance in which we suggest the use of a CAS or programmable calculator (see p. 789 of textbook for a list of software) for calculating some of the Fourier coefficients for Prob. 19 since they are quite involved. The purpose of the problem is to learn how to apply Fourier series.

Sec. 11.3 Forced Oscillations

The main point of this section is illustrated in Fig. 277, p. 494. It shows the unexpected reaction of a mass–spring system to a driving force that is periodic but not just a cosine or sine term. This reaction is explained by the use of the Fourier series of the driving force.

Problem Set 11.3. Page 494

7. Sinusoidal driving force. We have to solve the second-order nonhomogeneous ODE:

$$y'' + \omega^2 y = \sin t, \qquad \omega = 0.5, 0.9, 1.1, 1.5, 10.$$

The problem suggests the use of the method of undetermined coefficients of Sec. 2.7, pp. 81–84.

Step 1. General solution of homogeneous ODE

The ODE $y'' + \omega^2 y = 0$ is a homogeneous ODE with constant coefficients [see (1), p. 53]. It has the characteristic equation

$$\lambda^2 + \omega^2 = 0$$

so that

$$\lambda = \pm\sqrt{-\omega^2} = \pm\omega i.$$

This corresponds to Case III in the table on p. 58 of Sec. 2.2:

$$\begin{aligned} \lambda_1 &= -\tfrac{1}{2}a + i\omega = i\omega \quad (\text{since } a = 0), \\ \lambda_2 &= -i\omega. \end{aligned}$$

Hence

$$y_h = A\cos\omega t + B\sin\omega t \qquad (\text{where } e^{-at/2} = e^0 = 1).$$

Step 2. Solution of the nonhomogeneous ODE

We have to find the particular solution $y_p(t)$. By Table 2.1, p. 82,

$$\begin{aligned} y_p(t) &= K\cos t + M\sin t, \\ y_p'(t) &= -K\sin t + M\cos t, \\ y_p''(t) &= -K\cos t - M\sin t. \end{aligned}$$

Substituting this into the given ODE, we get

$$-K\cos t - M\sin t + \omega^2(K\cos t + M\sin t) = \sin t,$$

which, regrouped, is

$$-K\cos t + \omega^2 K\cos t - M\sin t + \omega^2 M\sin t = \sin t.$$

Comparing coefficients on both sides,

$$\begin{aligned} -M + \omega^2 M &= 1, \\ (\omega^2 - 1)M &= 1, \\ M &= \frac{1}{\omega^2 - 1} \quad \text{where} \quad |\omega| \neq 1. \end{aligned}$$

Also

$$\begin{aligned} -K + \omega^2 K &= 0, \\ K &= 0, \end{aligned}$$

so that together

$$y_p(x) = M \sin t = \frac{1}{\omega^2 - 1} \sin t.$$

Step 3. Solution of the given ODE

Putting it all together, we get the solution of our given ODE, that is,

$$y = y_h + y_p = A \cos \omega x + B \sin \omega x + \frac{1}{\omega^2 - 1} \sin t.$$

This corresponds to the solution given on p. A29.

Step 4. Consideration of different values of ω

Consider

$$a(\omega) = \frac{1}{\omega^2 - 1}.$$

Then

$$\begin{aligned} a(0.5) &= \frac{1}{0.5^2 - 1} = \frac{1}{-0.75} = -1.33, \\ a(0.9) &= \frac{1}{-0.19} = -5.26, \\ a(1.1) &= \frac{1}{0.21} = 4.76, \\ a(1.5) &= \frac{1}{1.25} = 0.80, \\ a(10) &= \frac{1}{99} = 0.0101. \end{aligned}$$

Step 5. Interpretation

Step 4 clearly shows that $a(\omega)$ becomes larger in absolute value the more closely we approach the point of resonance $\omega^2 = 1$. This motivates the different values of ω suggested in the problem. Note also that $a(\omega)$ is negative as long as $\omega\,(>0)$ is less than 1 and positive for values >1. This illustrates Figs. 54, p. 88, and 58, p. 91 (with $c = 0$), in Sec. 2.8, on modeling forced oscillations and resonance.

15. Forced undamped oscillations. The physics of this problem is as in Example 1, p. 492, and in greater detail on pp. 89–91 of Sec. 2.8. The driving force of the differential equation

$$y'' + cy' + y = r(t) = t(\pi^2 - t^2) \tag{I}$$

is odd and is of period 2π. Hence its Fourier series is a Fourier sine series, with the general term of the form

$$b_n \sin nt. \tag{II}$$

We shall determine b_n later in the solution. We want to determine a particular solution (I) (review pp. 81–84; see also pp. 85–91) with $r(t)$ replaced by (II), that is,

$$y'' + cy' + y = b_n \sin nt. \tag{III}$$

Just as in Example 1, pp. 492–493, we set

$$y = A \cos nt + B \sin nt. \tag{IV}$$

Note that, because the presence of the damping term cy' in the given ODE, we have both a cosine term and a sine term in (IV). We write y, A, B instead of y_n, A_n, B_n and thereby keep the formulas simpler. Next we calculate, from (IV),

$$\begin{aligned} y' &= -nA \sin nt + nB \cos nt, \\ y'' &= -n^2 A \cos nt - n^2 B \sin nt. \end{aligned}$$

We substitute y, y', y'' into (III). Write out the details and rearrange the terms by cosine and sine. We compare coefficients. The coefficients of the cosine terms must add up to 0 because there is no cosine term on the right side of (III). This gives

$$A + cnB - n^2 A = 0.$$

Similarly, the sum of the coefficients of the sine terms must equal the coefficient b_n on the right side of (III). This gives

$$B - cnA - n^2 B = b_n.$$

See that you get the same result. Ordering the terms of these two equations, we obtain

$$\begin{aligned} (1 - n^2)A + \quad cn \quad B &= 0, \\ -cn \quad A + (1 - n^2)B &= b_n. \end{aligned}$$

One can solve for A and B by elimination or by Cramer's rule (Sec. 7.6, p. 292). We use Cramer's rule. We need the following determinants. The coefficient determinant is

$$D = \begin{vmatrix} 1 - n^2 & cn \\ -cn & 1 - n^2 \end{vmatrix} = (1 - n^2)^2 + c^2 n^2. \tag{C}$$

The determinant in the numerator of the formula for A is

$$\begin{vmatrix} 0 & cn \\ b_n & 1 - n^2 \end{vmatrix} = -cnb_n.$$

The determinant in the numerator of the formula for B is

$$\begin{vmatrix} 1-n^2 & 0 \\ -cn & b_n \end{vmatrix} = (1-n^2)b_n.$$

Forming the ratios of the determinants, with the determinant of the coefficient matrix in the numerator, gives us the solution of the system of linear equations (see Example 1, p. 292). Hence

$$A = \frac{-cnb_n}{D}, \qquad B = \frac{(1-n^2)b_n}{D}.$$

We finally determine b_n from the second formula of the summary on the top of p. 487, and we get

$$\text{(V)} \qquad \begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx \qquad n = 1, 2, 3, \ldots \\ &= \frac{2}{\pi}\int_0^\pi t(\pi^2 - t^2)\sin nt\,dt \\ &= 2\pi\int_0^\pi t\ \sin nt\,dt - \frac{2}{\pi}\int_0^\pi t^3 \sin nt\,dt. \end{aligned}$$

The first integral (indefinite form) is solved by integration by parts. Throughout the integrations we shall ignore constants of integration.

$$\begin{aligned} \int t\sin nt\,dt &= -\frac{t}{n}\cos nt + \frac{1}{n}\int \cos nt\,dt \\ &= -\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt \end{aligned}$$

from which

$$\text{(VI)} \qquad \begin{aligned} 2\pi\int_0^\pi t\sin nt\,dt &= 2\pi\left[-\frac{t}{n}\cos nt\right]_0^\pi + 2\pi\left[-\frac{1}{n^2}\sin nt\right]_0^\pi \\ &= 2\pi\left(-\frac{\pi}{n}\cos n\pi\right) \\ &= -\frac{2\pi^2}{n}(-1)^n. \end{aligned}$$

The second integral is solved similarly. It requires repeated use of integration by parts.

$$\begin{aligned} \int t^3 \sin nt\,dt &= -\frac{t^3}{n}\cos nt + \frac{3}{n}\int t^2\cos nt\,dt \\ &= -\frac{t^3}{n}\cos nt + \frac{3}{n^2}t^2\sin nt - \frac{6}{n^2}\int t\sin nt\,dt \\ &= -\frac{t^3}{n}\cos nt + \frac{3}{n^2}t^2\sin nt - \frac{6}{n^2}\left(-\frac{t}{n}\ \cos nt + \frac{1}{n^2}\sin nt\right) \\ &= -\frac{t^3}{n}\cos nt + \frac{3}{n^2}t^2\sin nt + \frac{6}{n^3}t\ \cos nt - \frac{6}{n^4}\sin nt \end{aligned}$$

from which, making use of $\cos n\pi = (-1)^n$ and $\sin n0 = 0$,

$$\text{(VII)} \qquad -\frac{2}{\pi}\int_0^{\pi} t^3 \sin nt\, dt = \frac{2\pi^2}{n}(-1)^n - \frac{12}{n^3}(-1)^n.$$

Hence, from VI and VII,

$$\begin{aligned} b_n &= -\frac{2\pi^2}{n}(-1)^n + \frac{2\pi^2}{n}(-1)^n - \frac{12}{n^3}(-1)^n \\ &= -\frac{12}{n^3}(-1)^n \\ &= \frac{12(-1)^{n+1}}{n^3}. \end{aligned}$$

Hence

$$A = \frac{-cnb_n}{D} = \frac{-cn \cdot 12(-1)^{n+1}}{n^3 \cdot D} = \frac{cn \cdot 12(-1)^{n+2}}{n^3 \cdot D} = \frac{12c \cdot (-1)^n}{n^2 \cdot D}.$$

Similarly,

$$B = \frac{(1-n^2)12 \cdot (-1)^{n+1}}{n^3 \cdot D}$$

with D as given by (C) from before. This is in agreement with the steady-state solution given on p. A29 of App. 2 of the textbook. Putting it all together, we obtain the steady-state solution

$$y = \sum_{n=1}^{\infty}\left[\frac{12c \cdot (-1)^n}{n^2 \cdot D}\cos nt + \frac{(1-n^2)12 \cdot (-1)^{n+1}}{n^3 \cdot D}\sin nt\right]$$

where

$$D = (1-n^2)^2 + c^2n^2.$$

Sec. 11.4 Approximation by Trigonometric Polynomials

Approximations by trigonometric polynomials are suitable in connection with Fourier series as shown in **Example 1**, p. 497, and **Prob. 7**. Similarly, we shall see, in numerics in Secs. 19.3 and 19.4, that approximations by polynomials are appropriate in connection with Taylor series. **Parzeval's identity** (8), p. 497, is used to calculate the sum of a given series in **Prob. 11**.

Problem Set 11.4. Page 498

5. Minimum square error. The minimum square error E^* is given by (6), p. 496, that is,

$$E^* = \int_{-\pi}^{\pi} f^2\, dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right].$$

All of our computations revolve around (6).

To compute E^* we need $\int_{-\pi}^{\pi} f^2\,dx$. Here

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0, \\ 1 & \text{if } \quad 0 < x < \pi. \end{cases}$$

Hence

$$\int_{-\pi}^{\pi} f^2\,dx = \int_{-\pi}^{0} (-1)^2\,dx + \int_{0}^{\pi} 1^2\,dx = \big[x\big]_{-\pi}^{0} + \big[x\big]_{0}^{\pi} = 2\pi.$$

Next we need the Fourier coefficients of f. Since f is odd, the a_n are zero by (5**), p. 486, and Summary on p. 487. We compute the coefficients by the last formula of that Summary:

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} 1\cdot\sin nx\,dx \\ &= \frac{2}{n\pi}\big[-\cos n\pi\big]_0^{\pi} \\ &= -\frac{2}{n\pi}[(-1)^n - 1]. \end{aligned}$$

When $n = 2, 4, 6, \ldots$, then $(-1)^n - 1 = 1 - 1 = 0$ so that $b_n = 0$.
When $n = 1, 3, 5, \ldots$, then

$$b_n = -\frac{2}{n\pi}(-1-1) = \frac{4}{n\pi}.$$

Hence

$$b_n = \begin{cases} \dfrac{4}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

(This is similar to Example 1, pp. 477–478, in Sec. 11.1.)
By (2), p. 495,

$$\begin{aligned} F(x) &= A_0 + \sum_{n=1}^{N} (A_n\cos nx + B_n \sin nx) \\ &= \sum_{n=1}^{N} b_n \sin nx \\ &= \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots + \frac{1}{N}\sin Nx\right) \qquad \text{for } N \text{ odd.} \end{aligned}$$

Together we get

$$\begin{aligned} E^* &= 2\pi - \pi\left(\sum_{n=1}^{N} b_n^2\right) \\ &= 2\pi - \pi\left(\frac{16}{\pi^2}\right)\left(1 + 0 + \frac{1}{9} + 0 + \frac{1}{25} + \cdots + \frac{1}{N^2}\right) \qquad \text{for } N \text{ odd.} \end{aligned}$$

Complete the problem by computing the first five values for E^* and, if you use your CAS, compute the 20th value for E^*. Comment on the result and compare your answer with the one on p. A29 of the textbook.

9. Monotonicity of the minimum square error. The minimum square error E^* given by (6), p. 496, is monotone decreasing, that is, it cannot increase if we add further terms by choosing a larger N for the approximating polynomial. To prove this, we note that the terms in the sum in (6) are squares, hence they are nonnegative. Since the sum is subtracted from the first term in (6), which is the integral, the whole expression cannot increase. This is what is meant by "monotone decreasing," which, by definition, includes the case that an expression remains constant, in our case,

$$E_N^* \leq E_M^* \quad \text{if} \quad N > M,$$

where M and N are upper summation limits in (6).

11. Parseval's identity is given by (8), p. 497. It is

$$2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx.$$

We want to use (8) to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} = 1.233700550.$$

We are given the hint to use Example 1, pp. 477–478, of Sec. 11.1 It defines

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0, \\ k & \text{if } 0 < x < \pi, \end{cases}$$

and $f(x + 2\pi) = f(x)$ (i.e., f is periodic with 2π). First we compute the integral on the right-hand side of (8):

$$\int_{-\pi}^{\pi} f(x)^2 \, dx = \int_{-\pi}^{0} (-k)^2 \, dx + \int_{0}^{\pi} k^2 \, dx = \left[kx^2\right]_{-\pi}^{0} + \left[kx^2\right]_{0}^{\pi} = 2\pi k^2.$$

Hence the right-hand side of (8) is equal $2k^2$. From Example 2, p. 485, we also know that

$$b_1 = \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{4k}{5\pi}, \quad \cdots .$$

Also $a_0 = 0$ and $a_n = 0$ for $n = 1, 2, 3, \ldots$. Thus the left-hand side of (8) simplifies to

$$\begin{aligned} \sum_{n=1}^{\infty} b^2 &= b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + \cdots \\ &= b_1^2 + b_3^2 + b_5^2 + \cdots \\ &= \left(\frac{4k}{\pi}\right)^2 + \left(\frac{4k}{3\pi}\right)^2 + \left(\frac{4k}{5\pi}\right)^2 + \cdots \\ &= \frac{16k^2}{\pi^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right). \end{aligned}$$

Parzeval's identity gives us

$$\frac{16k^2}{\pi^2}\left(1+\frac{1}{9}+\frac{1}{25}+\cdots\right)=2k^2.$$

We multiply both sides by $\dfrac{\pi^2}{16k^2}$ and get

$$1+\frac{1}{9}+\frac{1}{25}+\cdots=\frac{2k^2\pi^2}{16k^2}=\frac{\pi^2}{8}.$$

You should check how fast the convergence is.

Sec. 11.5 Sturm–Liouville Problems. Orthogonal Functions

Sections 11.5 and 11.6 pursue a new idea of what happens when we replace the orthogonal trigonometric system of the Fourier series by other orthogonal systems. We define a **Sturm–Liouville problem,** p. 499, by an ODE of the form (1) with a parameter λ, and two boundary conditions (2) at two boundary points (endpoints) a and b and note that the problem is an example of a **boundary value problem**. We want to solve this problem by finding nontrivial solutions of (1) that satisfy the boundary conditions (2). These are solutions that are not identically zero. The solutions are *eigenfunctions* $y(x)$. The number λ (read "lambda," it is a Greek letter, standard notation!) for which such an eigenfunction exists, is an **eigenvalue** of the Sturm–Liouville problem. Thus the Sturm–Liouville problem is an **eigenvalue problem** for ODEs (see also chart on p. 323).

Example 1, p. 499, and **Prob. 9** solve a Sturm–Liouville problem. We define **orthogonality** and **orthonormality** for sets of functions on pp. 500–501. **Theorem 1**, p. 501, shows that, if we can formulate a problem as a Sturm–Liouville problem, then we are guaranteed orthogonality. Note that in Example 1 the cosine and sine functions constitute the simplest and most important set of orthogonal functions because, as we know, they lead to Fourier series—whose invention was perhaps the most important progress ever made in applied mathematics. Indeed, these series turned out to be fundamental for both ODEs and PDEs as shown in Chaps. 2 and 12.

Example 4 on p. 503 is remarkable inasmuch as no boundary conditions are needed when dealing with Legendre polynomials.

Problem Set 11.5. Page 503

1. Orthogonality of eigenfunctions of Sturm–Liouville problems. Supplying more details to the proof of Theorem 1, pp. 501–502.

Proof of Case 3. In this case the proof runs as follows. We assume Case 3, that is,

$$p(a)=0, \qquad p(b)\neq 0.$$

As before, the starting point of the proof is (9) on p. 502, which consists of two expressions, denoted in the textbook and here by "Line 1" and "Line 2." Since $p(a)=0$, we see that (9) reduces to

$$\text{(Line 1)} \qquad p(b)[\,y_n'(b)y_m(b)-y_m'(b)y_n(b)], \tag{9}$$

and we have to show that this expression is zero. Now from (2b), p. 499, we have

$$l_1 y_n(b)+l_2 y_n'(b)=0, \tag{B1}$$

$$l_1 y_m(b)+l_2 y_m'(b)=0. \tag{B2}$$

At least one of the two coefficients must be different from zero, by assumption, say,

$$l_2 \neq 0.$$

We multiply (B1) by $y_m(b)$ and (B2) by $-y_n(b)$ and add, thereby obtaining

$$l_2[\, y_n'(b)y_m(b) - y_m'(b)y_n(b)] = 0.$$

Since l_2 is not zero, the expression in the brackets must be zero. But this expression is identical with that in the brackets in (9) (Line 1). The second line of (9), p. 502, that is,

$$\text{(9)} \qquad \text{(Line 2)} \qquad -p(a)[\, y_n'(a)y_m(a) - y_m'(a)y_n(a)]$$

is zero because of the assumption $p(a) = 0$. Hence (9) is zero, and from (8), p. 502, we obtain the orthogonality relationship (6) of Theorem 1, which had to be proved. To complete the proof of Case 3, assume that

$$l_1 \neq 0$$

and proceed similarly. You should supply the details. This will help you learn the new material and show whether you really understand the present proof. Furthermore, following the lead on p. 502, develop the proof for **Case 4** $p(a) \neq 0$, $p(b) \neq 0$.

3. Change of x. To solve this problem, equate $ct + k$ to the endpoints of the given interval. Then solve for t to get the new interval on which you can prove orthogonality.

9. Sturm–Liouville problem. *Step 1. Setting up the problem*

The given equation

$$y'' + \lambda y = 0$$

with the boundary condition

$$y(0) = 0, \qquad y'(L) = 0$$

is indeed a Sturm–Liouville problem: The given equation is of the form (1), p. 499,

$$\text{(1)} \qquad [\, p(x)y'] + [q(x) + \lambda r(x)]y = 0$$

with $p = 1$, $q = 0$, $r = 1$. The interval in which solutions are sought is given by $a = 0$ and $b = L$ as its endpoints. In the boundary conditions

$$\text{(2a)} \qquad k_1 y + k_2 y' = 0 \qquad \text{at} \qquad x = a = 0,$$

$$\text{(2b)} \qquad l_1 y + l_2 y' = 0 \qquad \text{at} \qquad x = b = L,$$

where

$$k_1 = 1, \quad k_2 = 0, \quad l_1 = 0, \quad l_2 = 0.$$

Step 2. Finding a general solution

The given second-order linear ODE is of the form $y'' + ky = 0$ (with constant $k = \lambda$) and is solved by setting up the corresponding characteristic equation (3), p. 54, in Sec. 2.2, that is,

$$\tilde{\lambda}^2 + k = 0$$

so that

$$\tilde{\lambda} = \pm\sqrt{-k} = \pm\sqrt{-1}\sqrt{k} = \pm i\sqrt{k}.$$

This corresponds to Case III of Summary of Cases I–III on p. 58 of Sec. 2.2 with $-\left(\frac{1}{2}\right)a = 0$ so that $e^{-ax/2} = e^0 = 1$, $i\omega = i\sqrt{k}$ giving $\omega = \sqrt{k}$. Thus the general solution is

$$y(0) = A\cos kx + B\sin kx \quad \text{where} \quad k = \sqrt{\lambda}$$

so that

$$\text{(C)} \qquad y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

Step 3. Find the eigenvalues and eigenfunctions

We obtain the eigenvalues and eigenfunctions by using the boundary conditions. We follow Example 5, p. 57, of the textbook. The first initial condition, $y = 0$ at $x = a = 0$, substituted into (C) gives

$$\begin{aligned} y(0) &= A\ \cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \\ &= A\cos 0 + B\sin 0 = A\cdot 1 + 0 \\ \text{(D)} \qquad &= A = 0. \end{aligned}$$

(D) into (C) gives

$$\text{(E)} \qquad y = B\sin\sqrt{\lambda}x.$$

Applying the chain rule to (E) yields

$$\text{(F)} \qquad y' = (B\cos\sqrt{\lambda}x)\sqrt{\lambda} = \sqrt{\lambda}B\cos\sqrt{\lambda}x.$$

We substitute the boundary condition $y'(L) = 0$ into (F) and get

$$\text{(G)} \qquad y'(L) = \sqrt{\lambda}B\cos\sqrt{\lambda}L = 0.$$

This gives the condition

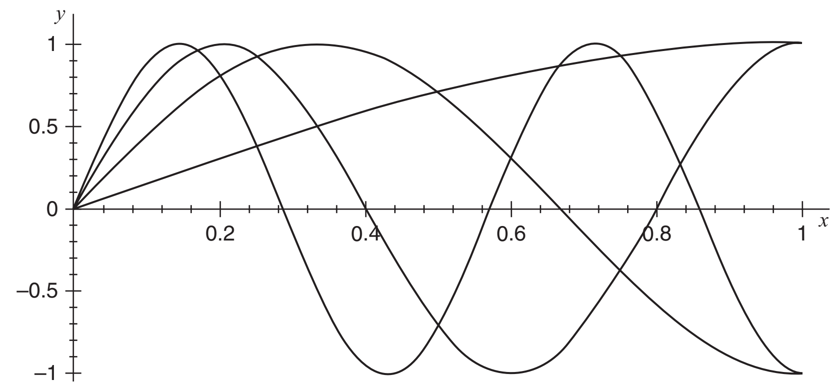
$$\text{(G}'\text{)} \qquad \cos\sqrt{\lambda}L = 0.$$

We know that $\cos z = 0$ for $z = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$ (odd). This means that

$$z = \pm\frac{(2n+1)\pi}{2} \qquad n = 0, 1, 2, 3, \ldots.$$

However, the cosine is an even function on the entire real axis, that is, $\cos(-z) = \cos z$, so we would get the same eigenfunctions. Hence

$$\cos\sqrt{\lambda}L = 0 \qquad \text{for} \qquad \sqrt{\lambda}L = \frac{(2n+1)\pi}{2} \qquad (n = 0, 1, 2, 3, \ldots).$$



Sec. 11.5 Prob. 9. First four eigenfunctions of the Sturm–Liouville problem

This gives us

$$\sqrt{\lambda} = \frac{(2n+1)\pi}{2L}.$$

Hence the eigenvalues are

$$\lambda_n = \left[\frac{(2n+1)\pi}{2L}\right]^2 \qquad (n = 0, 1, 2, 3, \ldots).$$

The corresponding eigenfunctions are

$$y(x) = y_n(x) = B \sin \sqrt{\lambda_n} x = B \sin \frac{(2n+1)\pi}{2L}.$$

If we choose $B = 1$, then

$$y(x) = y_n(x) = \sin \sqrt{\lambda_n} x = \sin \frac{(2n+1)\pi}{2L}.$$

Graph. The figure shows the first few eigenfunctions assuming that $L = 1$. All of them start at 0 and have a horizontal tangent at the other end of the interval from 0 to 1. This is the geometric meaning of the boundary conditions. y_1 has no zero in the interior of this interval. Its graph shown corresponds to $\frac{1}{4}$ of the period of the cosine. y_2 has one such zero (at $\frac{2}{3}$), and its graph shown corresponds to $\frac{3}{4}$ of that period. y_3 has two such zeros (at 0.4 and 0.8). y_4 has three, and so on.

Sec. 11.6 Orthogonal Series. Generalized Fourier Series

Here are ideas about Sec. 11.6 worth considering. In **Example 1**, pp. 505–506, the first two terms have the coefficients that are the largest in absolute value, much larger than those of any further terms, because $a_1P_1(x) + a_3P_3(x)$ resembles $\sin \pi x$ very closely. Make a sketch, using Fig. 107 on p. 178. Also notice that $a_2 = a_4 = \cdots = 0$ because $\sin \pi x$ is an odd function.

Example 2 and **Theorem 1**, both on p. 506, concern Bessel functions, infinitely many for every fixed n. So in (8), n is fixed. The smallest n is $n = 0$. Then (8) concerns J_0. Equation (8) now takes the form

$$\int_0^R x\, J_0(k_{m0}\, x) J_0(k_{j0}\, x)\, dx = 0 \quad (j \neq m, \text{ both integer}).$$

If $n = 0$ were the only possible value of n, we could simply write k_m and k_j instead of k_{m0} and k_{j0}; write it down for yourself to see what (8) then looks like. Recall that k_{m0} is related to the zero α_{m0} of J_0 by $k_{m0} = \alpha_{m0}/R$. In applications, R can have any value depending on the problem. For instance, in the vibrating drumhead, on pp. 586–590 in Sec. 12.10, the number R can have any value, depending on the problem. In Sec. 12.10, R is the radius of the drumhead. This is the reason for introducing the arbitrary k near the beginning of the example; it gives us the flexibility needed in practice.

Example 3, p. 507, shows a Fourier–Bessel series in terms of J_0, with arguments of the terms determined by the location of the zeros of J_0. The next series would be in terms of J_1, the second next in terms of J_2, and so on.

The last part of the section on pp. 507–509 concerns *mean square convergence*, a concept of convergence that is suitable in connection with orthogonal expansions and is quite different from the convergence considered in calculus. The basics are given in the section, and more details, which are not needed here, belong to special courses of a higher level; see, for instance, *Kreyszig's book on functional analysis* (see [GenRef7], p. A1 in App. 1).

Problem Set 11.6. Page 509

1. Fourier–Legendre series. In Example 1, on p. 505 of the text, we had to determine the coefficients by integration. In the present case this would be possible, but it is much easier to determine the coefficients directly by setting up and solving a system of linear equations as follows. The given function

$$f(x) = 63x^5 - 90x^3 + 35x$$

is of degree 5, hence we need only $P_0, P_1, P_2, \ldots, P_5$. Since f is an odd function, that is, $f(-x) = -f(x)$, we actually need only P_1, P_3, P_5. We write

$$f(x) = a_5P_5(x) + a_3P_3(x) + a_1P_1(x) = 63x^5 - 90x^3 + 35x. \tag{A}$$

Now from (11′), p. 178 in Sec. 5.2, we know that the Legendre polynomials P_1, P_3, P_5 are

$$\begin{aligned} P_5(x) &= \tfrac{1}{8}(63x^5 - 70x^3 + 15x), \\ P_3(x) &= \tfrac{1}{2}(5x^3 - 3x), \\ P_1(x) &= x. \end{aligned}$$

Substitution into (A) and writing it out yields

$$\begin{aligned} 63x^5 - 90x^3 + 35x = \tfrac{63}{8}a_5x^5 - \tfrac{70}{8}a_5x^3 &+ \tfrac{15}{8}a_5x \\ + \tfrac{5}{2}a_3x^3 &- \tfrac{3}{2}a_3x \\ &+ a_1x. \end{aligned}$$

The coefficients of the same power of x on both sides of the equation must be equal. This gives us the following system of equations:

$$\begin{aligned} 35\,x &= \tfrac{15}{8}a_5x - \tfrac{3}{2}a_3x + a_1x, \\ -90\,x^3 &= -\tfrac{70}{8}a_5x^3 + \tfrac{5}{2}a_3x^3, \\ 63\,x^5 &= \tfrac{63}{8}a_5x^5. \end{aligned}$$

Comparing the coefficients we obtain a linear system in a_1, a_3, and a_5, which we can write as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & \frac{15}{8} & 35 \\ 0 & \frac{5}{2} & -\frac{70}{8} & -90 \\ 0 & 0 & \frac{63}{8} & 63 \end{array}\right].$$

The augmented matrix is already in echelon form, so we are just going to do the back substitution step (see p. 276 of the textbook):

$$\tfrac{63}{8}a_5 = 63, \quad \text{hence} \quad a_5 = \tfrac{8}{63}\cdot 63 = 8.$$

Then

$$\tfrac{5}{2}a_3 - \tfrac{70}{8}\cdot 8 = -90, \quad \text{hence} \quad a_3 = \tfrac{2}{5}\cdot\tfrac{70}{8} - 90\cdot\tfrac{2}{5} = 2\cdot 14 - 18\cdot 2 = -8.$$

And finally

$$a_1 - \tfrac{3}{2}\cdot(-8) + \tfrac{15}{8}\cdot 8 = 35, \quad \text{hence} \quad a_1 + 12 + 15 = 35 \quad \text{so that} \quad a_1 = 35 - 27 = 8.$$

Hence

$$\begin{aligned} f(x) &= a_5P_5(x) + a_3P_3(x) + a_1P_1(x) \\ &= 8P_5(x) - 8P_3(x) + 8P_1(x) \\ &= 8[P_5(x) - P_3(x) + P_1(x)], \end{aligned}$$

which corresponds to the answer on p. A29.

5. Fourier–Legendre series. Even function. The Legendre polynomials $P_m(x)$, with odd m, contain only odd powers (see (11′), p. 178, Sec. 5.2). Hence P_m is an odd function, that is, $P_m(-x) = -P_m(x)$ because the sum of odd functions is odd. You should show this for our problem, noting that the odd functions consist of odd powers of x multiplied by nonzero constants.

Consider (3), p. 505,

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)\,P_m(x)\,dx. \tag{3}$$

Let f be even, that is, $f(-x) = f(x)$. Consider a_m in (3) with m an odd natural number. Denote the corresponding integrand by g, that is, $g(x) = f(x)P_m(x)$. The function g is odd, since

$$\begin{aligned} g(-x) &= f(-x)P_m(-x) \\ &= f(x)P_m(-x) && \text{(since } f \text{ is even)} \\ &= f(x)[-P_m(x)] && \text{(since } m \text{ is odd, } P_m \text{ is odd)} \\ &= -f(x)P_m(x) \\ &= -g(x). \end{aligned}$$

Let $w = -x$. Then $dw = -dx$ and $w = -1 \cdots 0$ corresponds to $x = 1 \cdots 0$. Hence

$$\begin{aligned}\int_{-1}^{0} g(w)\,dw &= \int_{1}^{0} g(-x)(-dx) \\ &= \int_{0}^{1} g(-x)\,dx \qquad \text{(minus sign absorbed by switching limits of integration)} \\ &= \int_{0}^{1} -g(x)\,dx \qquad \text{(since } g \text{ is an odd function)} \\ &= -\int_{0}^{1} g(x)\,dx.\end{aligned}$$

This shows that the integral of g from -1 to 0 is equal to (-1) times the integral of g from 0 to 1, so that the integral of g over the entire interval, that is, from -1 to 1 is zero. Hence $a_m = 0$.

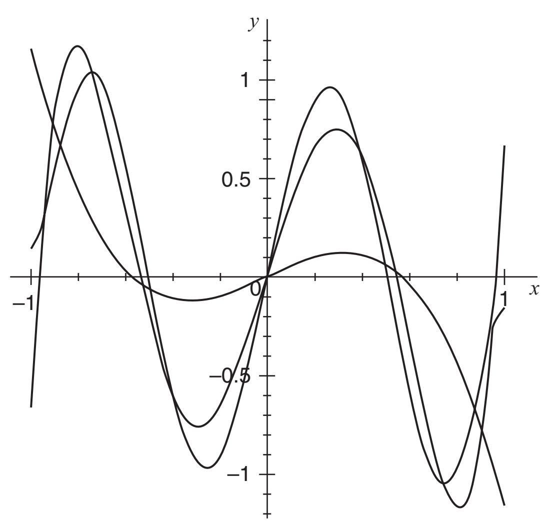
9. Fourier–Legendre series. The coefficients are given by (3), p. 505, in the form

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} (\sin 2\pi x)\, P_m(x)\, dx.$$

For even m these coefficients are zero (why?). For odd m the answer to this CAS experiment will have the coefficients

$$-0.4775, \quad -0.6908, \quad 1.844, \quad -0.8236, \quad \ldots,$$

which decrease rather slowly. Nevertheless, it is interesting to see, in the figure below, that convergence to $f(x) = \sin 2\pi x$ seems rather rapid. The figure shows the partial sums up to and including P_3, P_5, P_7. Explore with your CAS and see whether you can get a similar graph. Keep going until you find S_{m_0}. What is m_0?



Sec. 11.6 Prob. 9. Partial sums as mentioned for Fourier–Legendre series of $f(x) = \sin 2\pi x$

Sec. 11.7 Fourier Integral

Overview. The **Fourier integral** is given by (5), p. 513, with $A(w)$ and $B(w)$ given in (4). The integral is made plausible by Example 1 and the discussion on pp. 507–508. **Problem 1** solves such an integral.

Theorem 1 states sufficient conditions for the existence of a Fourier integral representation of a given function $f(x)$.

Example 2 shows an application leading to the so-called *sine integral* $\mathrm{Si}(x)$ given by (8) [we write $\mathrm{Si}(u)$ since x is needed otherwise], which cannot be evaluated by the usual methods of calculus.

For an even or odd function, the Fourier integral (5) becomes a Fourier cosine integral (10), p. 515, or a Fourier sine integral (11), respectively. These are applied in Example 3, **Prob. 7** and **Prob. 19.**

Problem Set 11.7. Page 517

1. Fourier integral.

First solution. *Working in the real. Integration by parts.* If only the integral were given, the problem would be difficult. The function on the right gives the idea of how we should proceed. The function is zero for negative x. For $x = 0$ it is $\pi/2$, which is the mean value of the limits from the left and right as x approaches 0. Essential to us is that $f(x) = \pi e^{-x}$ for $x > 0$. We use (4), p. 513. π cancels, and we have to integrate from 0 to ∞ because $f(x)$ is zero for negative x. Thus

$$A = \int_0^\infty e^{-v} \cos wv \, dv.$$

This integral can be solved as follows.

From calculus, using integration by parts, we get

$$\begin{aligned}\int e^{-x} \cos kx \, dx &= -e^{-x} \cos kx - \int (-k)(\sin kx)(e^{-x}) \, dx \\ &= -e^{-x} \cos kx + k \int (\sin kx)(e^{-x}) \, dx.\end{aligned}$$

Integration by parts on the last integral gives

$$\begin{aligned}\int (\sin kx)(e^{-x}) \, dx &= -e^{-x} \sin kx - \int k(\cos kx)(e^{-x}) \, dx \\ &= -e^{-x} \sin kx - k \int (\cos kx)(e^{-x}) \, dx.\end{aligned}$$

Thus we have two integrals that are the same (except for a constant).

$$\int e^{-x} \cos kx \, dx = -e^{-x} \cos kx - ke^{-x} \sin kx - k^2 \int (\cos kx)(e^{-x}) \, dx.$$

Adding the second integral to the first, one gets

$$(1 + k^2) \int e^{-x} \cos kx \, dx = -e^{-x} \cos kx - ke^{-x} \sin kx = e^{-x}(-\cos kx + k \sin kx).$$

Hence

$$\int e^{-x} \cos kx \, dx = \frac{e^{-x}}{1 + k^2}(-\cos kx + k \sin kx) + C.$$

The (definite) integral A is

$$\begin{aligned}\int_0^\infty e^{-v} \cos wv \, dv &= \left[\frac{e^{-v}}{1 + w^2}(-\cos wv + w \sin wv) \right]_0^\infty \\ &= \frac{1}{1 + w^2}\end{aligned}$$

since

$$\lim_{v\to\infty} \frac{e^{-v}}{1+w^2} = \lim_{v\to\infty} \frac{1}{e^{v}(1+w^2)} = 0$$

and the lower limit evaluates to

$$\frac{e^{-0}}{1+w^2}(-1+0) = \frac{-1}{1+w^2}.$$

Similarly, also from (4), p. 513, and evaluated in a likewise fashion

$$B = \int_0^\infty e^{-v} \sin wv \, dv = \left[\frac{e^{-v}}{1+w^2}(-\cos wv + w \sin wv)\right]_0^\infty = \frac{w}{1+w^2}.$$

Substituting A and B into (5), p. 513, gives the integral shown in the problem.

Second solution. *Working in the complex.* Integration by parts can be avoided by working in complex. From (4), using $\cos wv + i \sin wv = e^{iwv}$, we obtain (with a factor -1 resulting from the evaluation at the lower limit)

$$\begin{aligned} A + iB &= \int_0^\infty e^{-(v-iwv)} dv \\ &= \frac{1}{-(1-iw)} e^{-(1-iw)v}\Big|_0^\infty \\ &= \frac{1}{1-iw} = \frac{1+iw}{1+w^2}, \end{aligned}$$

where the last expression is obtained by multiplying numerator and denominator by $1 + iw$. Separation of the real and imaginary parts on both sides gives the integrals for A and B on the left and their values on the right, in agreement with the previous result.

7. Fourier cosine integral representation. We calculate A by (10), p. 515, and get [noting that $f(x) = 0$ for $x > 1$, thus the upper limit of integration is 1 instead of ∞]

$$\begin{aligned} A(w) &= \frac{2}{\pi} \int_0^1 f(v) \cos wv \, dv \\ &= \frac{2}{\pi} \int_0^1 1 \cdot \cos wv \, dv \\ &= \frac{2}{\pi} \left[\frac{\sin wv}{w}\right]_0^1 \\ &= \frac{2}{\pi} \frac{\sin w}{w}. \end{aligned}$$

Hence the Fourier integral is

$$\begin{aligned} f(x) &= \int_0^\infty A(w) \cos wx \, dw \\ &= \int_0^\infty \frac{2}{\pi} \frac{\sin w}{w} \cos wx \, dw \\ &= \frac{2}{\pi} \int_0^\infty \frac{\sin w}{w} \cos wx \, dw. \end{aligned}$$

19. Fourier sine integral representation. We calculate B by (11), p. 515, and get

$$B(w) = \frac{2}{\pi}\int_0^1 e^v \sin wv\, dv.$$

Let us solve the integral, using a more convenient notation from calculus. Integration by parts gives us

$$\int e^x \sin kx\, dx = e^x \sin kx - \int k(\cos kx)(e^x)\, dx.$$

Another integration by parts on the second integral:

$$\int e^x \cos kx\, dx = e^x \cos kx - \int (-k)(\sin kx)(e^x)\, dx.$$

Putting it together

$$\int e^x \sin kx\, dx = e^x \sin kx - k\left[e^x \cos kx + k\int (\sin kx)(e^x)\, dx\right].$$

Writing it out

$$\int e^x \sin kx\, dx = e^x \sin kx - ke^x \cos kx - k^2\int (\sin kx)(e^x)\, dx.$$

Taking the two integrals together

$$(1+k^2)\int e^x \sin kx\, dx = e^x \sin kx - ke^x \cos kx.$$

Thus

$$\int e^x \sin kx\, dx = \frac{e^x \sin kx - ke^x \cos kx}{1+k^2} + C.$$

In the original notation we have

$$\begin{aligned}\int_0^1 e^v \sin wv\, dv &= \left[\frac{e^v \sin wv - we^v \cos wv}{1+w^2}\right]_0^1 \\ &= \frac{e \sin w - we \cos w}{1+w^2} + \frac{w}{1+w^2}.\end{aligned}$$

Hence by (11), p. 515, the desired Fourier sine integral is

$$\begin{aligned}f(x) &= \int_0^\infty B(w) \sin wx\, dw \\ &= \frac{2}{\pi}\int_0^\infty \frac{1}{1+w^2}[e(\sin w - w\cos w) + w]\sin wx\, dw,\end{aligned}$$

which corresponds to the answer on p. A30 in App. 2 of the textbook.

Sec. 11.8 Fourier Cosine and Sine Transforms

Fourier transforms, p. 518, are the second most important transforms after Laplace transforms in Chap. 6, pp. 203–253. For even functions we get the related **Fourier cosine transforms** (1a), p. 518, illustrated in Example 1, p. 519, and Prob. 1. Similarly, for odd functions we develop **Fourier sine transforms** (2a), p. 518, shown in Example 1, p. 519, and Prob. 11. Other topics are inverse Fourier cosine and sine transforms, linearity, and derivatives, similar to Laplace transforms.

Indeed, the student who has studied the Laplace transforms will notice similarities with the Fourier cosine and sine transforms. In particular, formulas (4a,b), p. 520, and (5a,b), p. 521, have their counterpart in (1) and (2), p. 211.

Important Fourier cosine and sine transforms are tabulated in ***Sec. 11.10****, pp. 534–535.*

Problem Set 11.8. Page 522

1. Fourier cosine transform. From (1a), p. 518, we obtain (sketch the given function if necessary)

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

so that, by definition of f, noting that $f = 0$ for $x > 2$

$$\begin{aligned}
&= \sqrt{\frac{2}{\pi}} \left[\int_0^1 \cos wx \, dx + \int_1^2 (-1) \cos wx \, dx \right] \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{\sin wx}{w} \bigg|_0^1 - \frac{\sin wx}{w} \bigg|_1^2 \right) \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{\sin w}{w} - \frac{\sin 2w}{w} + \frac{\sin w}{w} \right) \\
&= \sqrt{\frac{2}{\pi}} \left(2\frac{\sin w}{w} - \frac{\sin 2w}{w} \right).
\end{aligned}$$

11. Fourier sine transform. We are given

$$f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1, \\ 0 & \text{if } x > 0. \end{cases}$$

The Fourier sine transform given by (2a), p. 518, is

$$\begin{aligned}
\hat{f}_s(w) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^1 x^2 \sin wx \, dx.
\end{aligned}$$

Tabular integration by parts. Consider $\int x^2 \sin wx \, dx$. It requires two successive integration by parts. To do so we set up the following table:

a. Differentiate the first column repeatedly as we go down. We start with x^2. Differentiate $(x^2)' = 2x$. Differentiate $(2x)' = 2$, etc.

b. Integrate the second column repeatedly as we go down. Integrate $\int \sin wx\,dx = -\frac{\cos wx}{w}$. Integrate $\int \left(-\frac{\cos wx}{w}\right) dx = -\frac{\sin wx}{w^2}$, etc.

c. We stop the process, when the derivative is 0. This happens because $(2)' = 0$.

d. Multiply the entries diagonally as indicated by the arrows. Alternate the signs of the resulting products, that is, $+ - + - \ldots$. Thus we have the terms

$$+\left[(x^2)\cdot\left(-\frac{\cos wx}{w}\right)\right] = -x^2\frac{\cos wx}{w}; \qquad -\left[(2x)\cdot\left(-\frac{\sin wx}{w^2}\right)\right] = 2x\frac{\sin wx}{w^2}, \text{ etc.}$$

$$\begin{array}{ccc}
\dfrac{d}{dx}\cdots & & \int \ldots dx \\
\text{Derivatives with respect to } x & & \text{Integrals with respect to } x \\
x^2 & & \sin \omega x \\
 & \searrow + & \\
2x & & -\dfrac{\cos \omega x}{\omega} \\
 & \searrow - & \\
2 & & -\dfrac{\sin \omega x}{\omega^2} \\
 & \searrow + & \\
0 & & \dfrac{\cos \omega x}{\omega^3}
\end{array}$$

Working through the tabular integration by parts, we obtain

$$\int x^2 \sin wx\,dx = -x^2\frac{\cos wx}{w} + 2x\frac{\sin wx}{w^2} + 2\frac{\cos wx}{w^3}.$$

Hence

$$\int_0^1 x^2 \sin wx\,dx = -\frac{\cos w}{w} + 2\frac{\sin w}{w^2} + 2\frac{\cos w}{w^3} - \frac{2}{w^3},$$

where the last term comes from $\cos 0 = 1$. Hence

$$\begin{aligned}
\hat{f}_s(w) &= \sqrt{\frac{2}{\pi}}\left(-\frac{\cos w}{w} + 2\frac{\sin w}{w^2} + 2\frac{\cos w}{w^3} - \frac{2}{w^3}\right) \\
&= \sqrt{\frac{2}{\pi}}\int_0^1 x^2 \sin wx\,dx
\end{aligned}$$

from which, by taking the common denominator of w^3, compares to the answer on p. A30. The tabular approach, where applicable, is much simpler and safer than the standard approach.

Can you identify other places in this Manual where we could have used tabular integration, e.g., in Prob. 15, Sec. 11.3 on p. 214?

Sec. 11.9 Fourier Transform. Discrete and Fast Fourier Transforms

This section concerns four related topics:

1. Derivation of the Fourier transform (6), p. 523, which is complex, from the complex Fourier integral (4), the latter being obtained by Euler's formula (3) and the trick of adding the integral (2) that is zero (pp. 518–519).

2. The physical aspect of the Fourier transform and spectral representations on p. 525.

3. Operational properties of the Fourier transform on pp. 526–528.

4. Representation of sampled values by the discrete Fourier transform on pp. 528–532 and **Prob. 19.** Computational efficiency leads from **discrete Fourier transform** (18), p. 530, **Example 4** to **fast Fourier transform** on pp. 531–532 and **Example 5.**

Problem Set 11.9. Page 533

3. Fourier transforms by integration. Calculation of Fourier transforms amounts to evaluating the defining integral (6), p. 523. For the function in Prob. 3 this simply means the integration of a complex exponential function, which is formally the same as in calculus. We integrate from a to b, the interval in which $f(x) = 1$, whereas it is zero outside this interval. According to (6) we obtain

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_a^b 1 \cdot e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{-iw} \right]_{x=a}^{b} = \frac{1}{-iw\sqrt{2\pi}}(e^{-iwb} - e^{-iwa}).$$

Now

$$\frac{1}{-i} = \frac{1}{-i} \cdot \frac{i}{i} = \frac{i}{-i^2} = \frac{i}{-(-1)} = i.$$

Hence

$$\hat{f}(w) = \frac{i}{w\sqrt{2\pi}}(e^{-iwb} - e^{-iwa}) \quad \text{if} \quad a < b \qquad \text{and} \qquad 0 \qquad \text{otherwise.}$$

15. Table III in Sec. 11.10 contains formulas of Fourier transforms, some of which are related. For deriving formula 1 from formula 2 start from formula 2 in Table III, p. 536, which lists the transform

$$\frac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$$

of the function *1* if $b < x < c$ and 0 otherwise. Since we need *1* for $-b < x < b$, it is clear that we should set $+b$ instead of $-b$ in the first term, and $c = b$ in the second term. We obtain

$$\text{(A)} \qquad \frac{e^{ibw} - e^{-ibw}}{iw\sqrt{2\pi}}.$$

Euler's formula (3), p. 523, states that

$$e^{ix} = \cos x + i \sin x.$$

Setting $x = bw$ in Euler's formula gives us

$$e^{ibw} = \cos bw + i \sin bw.$$

Setting $x = -bw$ in Euler's formula yields

$$e^{-ibw} = \cos(-bw) + i \sin(-bw) = \cos bw - i \sin bw$$

since cos is even and sin is odd. Hence the numerator of (A) simplifies to

$$\text{(B)} \qquad \begin{aligned} e^{ibw} - e^{-ibw} &= \cos bw + i \sin bw - (\cos bw - i \sin bw) \\ &= 2i \sin bw. \end{aligned}$$

Substituting (B) into (A) and algebraic simplification gives us

$$\frac{e^{ibw} - e^{-ibw}}{iw\sqrt{2\pi}} = \frac{2i \sin bw}{iw\sqrt{2\pi}} = \frac{2 \sin bw}{w\sqrt{2\pi}} = \frac{2}{\sqrt{2}\sqrt{\pi}} \frac{\sin bw}{w} = \sqrt{\frac{2}{\pi}} \frac{\sin bw}{w},$$

which is precisely the right side of formula 1 in Table III, p. 536, as wanted.

19. Discrete Fourier transform. The discrete Fourier transform of a given signal is

$$\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$$

where $\mathbf{F}_N$ is an $N \times N$ Fourier matrix whose entries are given by (18), p. 530. In our problem the given digital signal is

$$\hat{\mathbf{f}} = [f_1 \quad f_2 \quad f_3 \quad f_4]^\mathsf{T}.$$

Furthermore, here $\mathbf{F}_N$ is a 4×4 matrix and is given by (20), p. 530. Thus

$$\hat{\mathbf{f}} = F_4 \mathbf{f} = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

where

$$\begin{aligned} w &= e^{-2\pi i/4} = e^{-\pi i/2} \\ &= \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) \qquad \text{(by Euler's formula)} \\ &= 0 - i \\ &= -i. \end{aligned}$$

The entries in $\mathbf{F}_4$ are

$$\begin{aligned} w^0 &= 1, \\ w^1 &= -i, \\ w^2 &= (-i)^2 = i^2 = -1, \\ w^3 &= w^2 \cdot w = (-1)(-i) = i, \\ w^4 &= w^2 \cdot w^2 = (-1)(-1) = 1, \\ w^6 &= w^3 \cdot w^3 = i \cdot i = -1, \\ w^9 &= w^6 \cdot w^3 = (-1) \cdot i = -i. \end{aligned}$$

Thus

$$\hat{\mathbf{f}} = \mathbf{F}_4 \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} f_1 + f_2 + f_3 + f_4 \\ f_1 - if_2 - f_3 + if_4 \\ f_1 - f_2 + f_3 - f_4 \\ f_1 + if_2 - f_3 - if_4 \end{bmatrix}.$$

Note that we verified the entries of $\mathbf{F}_4$ by our calculations of exponents of w.

Chap. 12 Partial Differential Equations (PDEs)

Partial differential equations arise when the functions underlying physical problems depend on two or more independent variables. Usually, these variables are time t and one or several space variables. Thus, more problems can be modeled by PDEs than ODEs. However, your knowledge of ODEs is important in that solving PDEs may lead to systems of ODEs or even just one ODE. Many applications in fluid mechanics, elasticity, heat transfer, electromagnetic theory, quantum mechanics, and other areas of physics and engineering lead to partial differential equations.

Thus, it is important the engineer and physicist learn about important PDEs, such as the one-dimensional wave equation (Secs. 12.2–12.4, 12.12), the heat equation (Secs. 12.5–12.7), the two-dimensional wave equation (Secs. 12.8–12.10), and the Laplace equation (Sec. 12.11). The beauty of this chapter is that knowledge of previous areas such as ODEs, Fourier analysis, and others come together and show that engineering mathematics is a science with powerful unifying principles, as discussed in the third theme of the underlying themes on p. ix of the textbook.

For this chapter, you should, most importantly, know how to apply Fourier analysis (Chap. 11), know how to solve linear ODEs (Chap. 2), recall the technique of separating variables (Sec. 1.3), have some knowledge of Laplace transforms (only for Sec. 12.12), and, from calculus, know integration by parts and polar coordinates.

Since this chapter is quite involved and technical, make sure that you schedule sufficient study time. We go systematically and stepwise from modeling PDEs to solving them. This way we separate the derivation process of the PDEs from the solving process. This is shown in the **chapter orientation table** on the next page, which summarizes the main topics of this chapter and is organized by PDEs, modeling tasks (MTs), and solution methods (SMs).

Sec. 12.1 Basic Concepts of PDEs

We introduce basic concepts of PDEs (pp. 540–541) and list some very important PDEs ((1)–(6)) on p. 541. They are the *wave equation* in 1D, 2D (1), (5), the *heat equation* in 1D (2), the *Laplace equation* in 2D, 3D (3), (6) and the *Poisson equation* in 2D (4).

Many concepts from ODEs carry over to PDEs. They include order, linear, nonlinear, homogeneous, nonhomogeneous, boundary value problems (**Prob. 15**, p. 542), and others. However, there are differences when dealing with PDEs. A PDE has a much greater variety of solutions than an ODE. Whereas the solution of an ODE of second order contains two arbitrary *constants*, the solution of a PDE of second order generally contains two arbitrary *functions*. These functions need to be determined from initial conditions, which are given initial functions, say, given initial displacement and velocity of a mechanical system.

Finally, we explain the first method on how to solve a PDE. In the special case where a PDE involves derivatives with respect to one variable only (**Example 2**, p. 542, **Probs. 17** and **19**, p. 543) or can be transformed to such a form (**Example 3**, p. 542), we set up the corresponding ODE, solve it by methods of Part A of the textbook, and then use that solution to write down the general solution for the PDE.

Problem Set 12.1. Page 542

15. Boundary value problem. Problem 15 is similar to **Probs. 2–13** in that we verify solutions to important PDEs selected from (1)–(6), p. 541. Here we verify a boundary value problem instead of an individiual solution to PDE (3), p. 541.

The verification of the solution is as follows. Observing the chain rule, we obtain, by differentiation,

$$u_x = \frac{2ax}{x^2 + y^2}$$

Summary of content of Chap. 12

PDEs	Section	Modeling Tasks (MTs)/ Solution Methods (SMs)
General	**12.1** pp. 540–543	*SMs:* separating variables, treat special PDEs as ODEs
1D wave equation (3), p. 545	**12.2** pp. 543–545	*MTs:* vibrating string (e.g., violin string)
	12.3 pp. 545–553	*SMs:* separating variables, Fourier series, eigenfunctions
	12.4 pp. 553–556	*SMs:* D'Alembert's solution method of characteristics, pp. 555–556
Heat equation (3), p. 558	**12.5** pp. 557–558	*MTs:* temperature in a (finite) body
1D heat equation (1), p. 559	**12.6** pp. 558–567	*MTs:* temperature in long thin metal bar, Fig. 294, p. 559 *SMs:* separating variables, Fourier series, eigenfunctions, pp. 559–561
2D heat equation (14), p. 564 (Laplace's equation)		*MTs:* Dirchlet problem in a rectangle, Fig. 296, p. 564 *SMs:* separating variables, eigenfunctions, Fourier series, pp. 564–566
1D heat equation (1), p. 568	**12.7** pp. 568–574	*MTs:* temperature in very long "infinite" bar or wire, p. 568 *SMs:* Fourier integrals pp. 569–570 *SMs:* Fourier transforms, convolutions, pp. 571–574
2D wave equation (3), p. 577	**12.8** pp. 575–577	*MTs:* vibrating membrane, Fig. 301, p. 576
2D wave equation (1), p. 577	**12.9** pp. 577–585	*MTs:* vibrating rectangular membrane, see Fig. 302, p. 577 *SMs:* separating variables, p. 578 (2D Helmholtz equation, p. 578), eigenfunctions, double Fourier series, p. 582
2D wave equation in polar coordinates (6), p. 586	**12.10** pp. 585–592	*MTs:* vibrating circular membrane, Fig. 307, p. 586 *SMs:* Bessel's ODE (12), p. 587, Bessel functions, pp. 587–588, eigenfunctions, p. 588, Fourier–Bessel series, p. 589
Laplace's equation (1), p. 593 **Laplacian in** . . . . . . **cylindrical coordinates** (5), p. 594 . . . **spherical coordinates** (7), p. 594	**12.11** pp. 593–600	*MTs:* potential within a sphere, p. 596 potential outside a sphere, p. 597 spherical capacitor, Example 1, p. 597 and others (see problem set, p. 598) *SMs:* Dirchlet problem, separating variables, p. 595 Euler–Cauchy equation (13), p. 595, Legendre's equation (15′), p. 596, Fourier–Legendre series, p. 596
1D wave equation (1), p. 601	**12.12** pp. 600–603	*MTs:* vibrating semi-infinite string, p. 600 *SMs:* Laplace transforms, p. 601

and, by another differentiation, with the product rule applied to $2ax$ and $1/(x^2 + y^2)$,

(A)
$$u_{xx} = \frac{2\,a}{x^2 + y^2} + \frac{-1 \cdot 2ax \cdot 2x}{(x^2 + y^2)^2}.$$

Similarly, with y instead of x,

(B)
$$u_{yy} = \frac{2\,a}{x^2 + y^2} - \frac{4ay^2}{(x^2 + y^2)^2}.$$

Taking the common denominator $(x^2 + y^2)^2$, you obtain, in (A), the numerator

$$2a(x^2 + y^2) - 4ax^2 = -2ax^2 + 2ay^2$$

and, in (B),

$$2a(x^2 + y^2) - 4ay^2 = -2ay^2 + 2ax^2.$$

Addition of the two expressions on the right gives 0 and completes the verification.

Next, we determine a and b in $u(x, y) = a \ln(x^2 + y^2) + b$ from the boundary conditions. For $x^2 + y^2 = 1$, we have $\ln 1 = 0$, so that $b = 110$ from the first boundary condition. From this, and the second boundary condition $0 = a \ln 100 + b$, we obtain $a \ln 100 = -110$. Hence, $a = -110/\ln 100$, in agreement with the answer on p. A31.

17. PDE solvable as second-order ODE. We want to solve the PDE $u_{xx} + 16\pi^2 u = 0$.

Step 1. Verify that the given PDE contains only one variable and write the corresponding ODE.

In the given PDE y does not appear explicitly. Hence, we can solve this PDE like the ODE

(C)
$$u'' + 16\pi^2 u = 0.$$

Step 2. Solve the corresponding ODE.

We identify this as a homogeneous ODE (1), p. 53, in Sec. 2.2. To solve (C), we form the characteristic equation

$$\lambda^2 + 16\pi^2 = 0, \qquad \lambda = \pm\sqrt{-16\pi^2} = \pm 4\pi i.$$

From the Table of Cases I–III, p. 58, we have Case III, that is, $y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$. Here $a = 0$ and $\omega = 4\pi$ so that (A) has the solution

$$u = A \cos(4\pi x) + B \sin(4\pi x).$$

Step 3. Go back to PDE and write solution for the PDE.

Since we are dealing with a PDE, we look for a solution $u = u(x, y)$ so that we can have arbitrary functions $A = A(y)$ and $B = B(y)$. Hence, the solution to the PDE is

(D)
$$u(x, y) = A(y) \cos(4\pi x) + B(y) \sin(4\pi x).$$

Step 4. Check solution.

To show that (D) is a solution to the PDE we take partial derivatives of (D). We have (chain rule!)

$$\begin{aligned} u_x &= A(y)(-\sin(4\pi x)) \cdot 4\pi + B(y)(\cos(4\pi x)) \cdot 4\pi \\ &= -4\pi A(y) \sin(4\pi x) + 4\pi B(y) \cos(4\pi x). \end{aligned}$$

From this we take partial derivatives again and recognize u:

$$
\text{(E)} \qquad \begin{aligned}
u_{xx} &= -4\pi \cdot 4\pi A(y)\cos(4\pi x) - 4\pi \cdot 4\pi B(y)\sin(4\pi x) \\
&= -16\pi^2 A(y)\cos(4\pi x) - 16\pi^2 B(y)\sin(4\pi x) \\
&= -16\pi^2 (A(y)\cos(4\pi x) + B(y)\sin(4\pi x)) \\
&= -16\pi^2 u \qquad \text{[by (D)].}
\end{aligned}
$$

Hence,

$$u_{xx} + 16\pi^2 u = -16\pi^2(u) + 16\pi^2 u = 0 \qquad \text{[by last line of (E)].}$$

Can you immediately give the solution for $u_{yy} + 16\pi^2 u = 0$?

19. PDE solvable as first-order ODE.

Step 1. Identification of PDE as ODE. Write ODE.

PDE contains no x variable. Corresponding ODE is

$$\text{(F)} \qquad u' + y^2 u = 0.$$

Step 2. Solve the corresponding ODE.

$$
\begin{aligned}
u' &= -y^2 u, \\
\frac{du}{dy} &= -y^2 u, \\
\frac{du}{u} &= -y^2\, dy.
\end{aligned}
$$

This separation of variables (p. 12, Sec. 1.3) gives

$$
\begin{aligned}
\int \frac{1}{u}\, du &= -\int y^2\, dy, \\
\ln u &= -\frac{y^3}{3} + C, \qquad \text{where } C \text{ is a constant.}
\end{aligned}
$$

Solve for u by exponentiation:

$$
\begin{aligned}
e^{\ln u} &= e^{-y^3/3 + C}, \\
u &= e^{-y^3/3} e^C.
\end{aligned}
$$

Call $e^C = c$, where c is some constant, and get

$$u = ce^{-y^3/3}.$$

Step 3. Go back to PDE and write solution of PDE.

$$\text{(G)} \qquad u(x, y) = c(x)e^{-y^3/3}.$$

Step 4. Check answer.

You may want to check the answer, as was done in Prob. 17.

Sec. 12.2 Modeling: Vibrating String. Wave Equation

We introduce the process of modeling for PDEs by deriving a model for the vibrations of an elastic string. We want to find the deflection $u(x, t)$ at any point x and time t. Since the number of variables is *more than one* (i.e., two), we need a *partial* differential equation. Turn to p. 543 and **go over this section very carefully**, as it explains several important ideas and techniques that can be applied to other modeling tasks. We describe the experiment of plucking a string, identifying a function $u(x, t)$ with two variables, and making reasonable physical assumptions. Briefly, they are perfect homogeneity and elasticity as, say, in a violin string, negligible gravity, and negligible deflection.

Our derivation of the PDE on p. 544 starts with an **important principle**: We consider the forces acting on a ***small*** *portion* of the string (instead of the entire string). This is shown in Fig. 286, p. 543. This method is typical in mechanics and other areas. We consider the forces acting on that portion of the string and equate them to the forces of inertia, according to Newton's second law (2), p. 63 in Sec. 2.4. Go through the derivation. We obtain the one-dimensional wave equation (3), a linear PDE. *Understanding this section well carries the reward that a very similar modeling approach will be used in Sec. 12.8, p. 575.*

Sec. 12.3 Solution by Separating Variables. Use of Fourier Series

Separating variables is a general solution principle for PDEs. It reduces the PDE to several ODEs to be solved, two in the present case because the PDE involves two independent variables, x and time t. We will use this principle again, twice for solving the heat equation in Sec. 12.6, p. 558, and once for the two-dimensional wave equation in Sec. 12.9, p. 577.

Separating variables gives you infinitely many solutions, but separating alone would not be enough to satisfy all the physical conditions. Indeed, to completely solve the physical problem, we will use those solutions obtained in the separation as terms of a Fourier series (review Secs. 11.1–11.3, in particular p. 476 in Sec. 11.1 and Summary on p. 487 in Sec. 11.2), whose coefficients we find from the initial displacement of the string and from its initial velocity.

Here is a **summary of Sec. 12.3 in terms of the main formulas**. We shall use it in presenting our solutions to **Probs. 5, 7**, and **9**.

1. *Mathematical description of problem.* The general problem of modeling a vibrating string fastened at the ends, plucked and released, leads to a PDE (1), p. 545, the **one-dimensional wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $c^2 = T/\rho$. Furthermore, since the string is fastened at the ends $x = 0$ and $x = L$, we have two boundary conditions (2a) and (2b), p. 545:

$$\text{(a)} \quad u(0, t) = 0, \qquad \text{(b)} \quad u(L, t) = 0. \tag{2}$$

Also, the plucking and release of the string gives a deflection at time $t = 0$ ("initial deflection") and a velocity at $t = 0$, which is expressed as two initial conditions (3a) and (3b), p. 545:

$$\text{(a)} \quad u(x, 0) = f(x), \qquad \text{(b)} \quad u_t(x, 0) = g(x) \qquad 0 \le x \le L. \tag{3}$$

Formulas (1), (2), and (3) describe the mathematical problem completely.

2. *Mathematical solution of problem.* The solution to this problem is given by (12), p. 548, that is,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x. \tag{12}$$

Substituting (9), p. 547, into (12) gives

(A) $$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos \frac{n\pi}{L} t + B_n^* \sin \frac{n\pi}{L} t \right) \sin \frac{n\pi}{L} x,$$

and, from (3a),

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} \left(B_n \cos \frac{n\pi}{L} 0 + B_n^* \sin \frac{n\pi}{L} 0 \right) \sin \frac{n\pi}{L} x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x,$$

so that, by (6**), p. 486,

(14) $$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx, \qquad n = 1, 2, 3, \cdots.$$

Similarly, the B_n^*'s are chosen in such a way that $\frac{\partial u}{\partial t}\Big|_{t=0}$ becomes the Fourier sine series (15), p. 549, of $g(x)$, that is

(15) $$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx, \qquad n = 1, 2, 3, \cdots.$$

Problem Set 12.3. Page 551

5. Vibrating string. In this problem we are given that the initial velocity is 0, that is,

$$g(x) = 0$$

so that B_n^* becomes

(15) $$B_n^* = \frac{2}{cn\pi} \int_0^L 0 \cdot \sin \frac{n\pi x}{L} \, dx = \frac{2}{cn\pi} \int_0^L 0 \cdot dx = 0, \qquad n = 1, 2, 3, \cdots.$$

Together with $L = 1$ and $c = 1$, equation (A) (see beginning of this section) simplifies to

(A1) $$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos\, n\pi t) \sin\, n\pi x.$$

Now, since the given initial deflection is $k \sin 3\pi x$ with $k = 0.01$, we have

(B) $$u(x,0) = k \sin 3\pi x.$$

However, (B) is already in the form of a Fourier sine series of $f(x)$ (with $B_n = 0$ for all $n \neq 3$ and $B_3 = k$), so we do not have to compute anything for (14). Hence (A1) simplifies further to

(A2) $$\begin{aligned} u(x,t) &= (k \cos 3\pi t) \sin 3\pi x \\ &= (0.01 \cos 3\pi t) \sin 3\pi x, \end{aligned}$$

corresponding to the answer on p. A31.

7. String with given initial deflection. We use the same approach as in Prob. 5. The only difference is the value of the initial deflection, which gives us

$$(0 < x < 1) \qquad u(x, 0) = kx(1 - x),$$

so we use (14) to compute the coefficients B_n

$$\text{(B)} \qquad B_n = 2\int_0^1 kx(1 - x)\sin n\pi x\, dx.$$

This is the Fourier series of the initial deflection. We obtain it as the half-range expansion of period $2L = 2$ in the form of a Fourier sine series with the coefficients (6**), p. 486 of Sec. 11.2. We multiply out the integrand and get

$$kx(1 - x)\sin \pi x = (kx - kx^2)\sin \pi x = kx \sin n\pi x - kx^2 \sin n\pi x.$$

This leads us to consider the two indefinite integrals

$$\int x \sin n\pi x\, dx \qquad \text{and} \qquad \int x^2 \sin n\pi x\, dx.$$

From Prob. 11, Sec. 11.8 on **p. 228** of this Manual with $w = n\pi$, we obtain

$$\int x^2 \sin n\pi x\, dx = -x^2 \frac{\cos n\pi x}{n\pi} + 2x\frac{\sin n\pi x}{n^2\pi^2} + 2\frac{\cos n\pi x}{n^3\pi^3}.$$

Using the technique of tabular integration by parts, demonstrated in Prob. 11, we set up the following table:

$\frac{d...}{dx}$ Derivatives with respect to x		$\int ...dx$ Integrals with respect to x
x		$\sin n\pi x$
1	$+$	$-\frac{\cos n\pi x}{n\pi}$
0	$-$	$-\frac{\sin n\pi x}{n^2\pi^2}$

so that

$$\int x \sin n\pi x\, dx = -x\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}.$$

Substituting the limits of integration

$$\int_0^1 x \sin n\pi x\, dx = -1\frac{\cos n\pi}{n\pi} + \frac{\sin n\pi}{n^2\pi^2}$$
$$= -\frac{\cos n\pi}{n\pi}$$
$$= -\frac{(-1)^n}{n\pi},$$

$$\begin{aligned}\int_0^1 x^2 \sin n\pi x\,dx &= -1\frac{\cos n\pi}{n\pi} + 2\frac{\sin n\pi}{n^2\pi^2} + 2\frac{\cos n\pi}{n^3\pi^3} - 2\frac{\cos 0}{n^3\pi^3}\\ &= -\frac{\cos n\pi}{n\pi} + 2\frac{\cos n\pi}{n^3\pi^3} - 2\frac{1}{n^3\pi^3}\\ &= -\frac{(-1)^n}{n\pi} + 2\frac{(-1)^n}{n^3\pi^3} - 2\frac{1}{n^3\pi^3}.\end{aligned}$$

Hence (B) is

$$\begin{aligned}2k\left[-\frac{(-1)^n}{n\pi}\right] - 2k\left[-\frac{(-1)^n}{n\pi} + 2\frac{(-1)^n}{n^3\pi^3} - 2\frac{1}{n^3\pi^3}\right] &= (-2k+2k)\frac{(-1)^n}{n\pi} - 4k\frac{(-1)^n}{n^3\pi^3} + 4k\frac{1}{n^3\pi^3}\\ &= -4k\frac{(-1)^n}{n^3\pi^3} + 4k\frac{1}{n^3\pi^3}\\ &= 4k\left[\frac{1-(-1)^n}{n^3\pi^3}\right].\end{aligned}$$

The numerator of the fraction in the last line equals

$$4k[1-(-1)^n] = \begin{cases} 8k & \text{for odd } n,\\ 0 & \text{for even } n.\end{cases}$$

This gives you the Fourier coefficients

$$b_1 = \frac{8k}{\pi^3}, \qquad b_2 = 0, \qquad b_3 = \frac{8k}{27\pi^3}, \qquad b_4 = 0, \qquad b_5 = \frac{8k}{125\pi^3}, \qquad b_6 = 0, \qquad \cdots.$$

Using the same approach as in **Prob. 5** of this section, formula (A1) of Prob. 5 becomes here

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} (B_n \cos n\pi t)\sin n\pi x\\ &= \frac{8k}{\pi^3}\cos \pi t \sin \pi x + \frac{8k}{27\pi^3}\cos 3\pi t \sin 3\pi x + \frac{8k}{125\pi^3}\cos 5\pi t \sin 5\pi x + \cdots,\end{aligned}$$

as on p. A31 of the textbook, where $k = 0.01$.

9. Use of Fourier series. Problems 7–13 amount to the determination of the Fourier sine series of the initial deflection. Each term $b_n \sin nx$ is then multiplied by the corresponding $\cos nt$ (since $c = 1$; for arbitrary c it would be $\cos cnt$). The series of the terms thus obtained is the desired solution. For the "triangular" initial deflection in Prob. 9, we obtain the Fourier sine series on p. 490 of Example 6 in Sec. 11.2 with $L = 1$ and $k = 0.1$ and get

$$\frac{0.8}{\pi^2}\left(\sin \pi x - \frac{1}{9}\sin 3\pi x + \frac{1}{25}\sin 5\pi x - + \cdots\right).$$

Multiplying each $\sin((2n+1)\pi x)$ term by the corresponding $\cos((2n+1)\pi t)$, we obtain the answer on p. A31.

Sec. 12.4 D'Alembert's Solution of the Wave Equation. Characteristics

Turn to pp. 553–556 of the textbook as we discuss the salient points of Sec. 12.4. D'Alembert's ingenious idea was to come up with substitutions ("transformations") (2), p. 553. This allowed him to transform the more difficult PDE (1)—the wave equation—into the much simpler PDE (3). PDE (3) quickly solves to (4), p. 554. Note that to get from (1) to (3) requires several applications of the chain rule (1), p. 393.

Next we show that d'Alembert's solution (4) is related to the model of the string of Secs. 12.2 and 12.3. By including initial conditions (5), p. 554 (which are also (3), p. 545), we obtain solution (12). In addition, because of the boundary conditions (2), p. 545, the function f in (12) must be odd and have period $2L$.

D'Alembert's method of solving the wave equation is simple, as we saw in the first part of this section. This raises the question of whether and to what PDEs his method can be extended. An answer is given in the **method of characteristics** on pp. 555–556. It consists of solving **quasilinear** PDEs of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y). \tag{14}$$

These PDEs are linear in the highest derivatives but may be arbitrary otherwise. We set up a **characteristic equation** of (14), which is

$$Ay'^2 - 2By' + C = 0. \tag{15}$$

(Be aware of the minus sign in front of $2B$: we have $-2B$, not $+2B$!)

We identify what type of PDE (14) is. If the **discriminant** $AC - B^2 < 0$, then we obtain a **hyperbolic PDE**. If the discriminant $= 0$, then we have a **parabolic PDE**. Finally, if the discriminant > 0, then (14) is an **elliptic PDE.**

Look at the table on p. 556. Based on the discriminant, we use different substitutions ("transformations") and obtain the normal form, a PDE much simpler than (14). We solve the normal form.

D'Alembert's method of solving the wave equation is more elegant than that of Sec. 12.3 but restricted to fewer PDEs.

Example 1, p. 556, shows how the method of characteristics can be used to arrive at d'Alembert's solution (4), p. 554.

Problem 11 shows, in great detail, how to transform a PDE (14) of the parabolic type into its normal form and then how to solve it. In doing so, it also shows several times how to apply the chain rule to partials and sheds more light on how to get from (1) to (3) on p. 553.

Problem Set 12.4. Page 556

1. Speed. This is a uniform motion (motion with constant speed). Use that, in this case, speed is distance divided by time or, equivalently, speed is distance traveled in unit time. Equivalently, $\partial(x + ct)/\partial t = c$.

11. Normal form.

Step 1. For the given PDE identify A, B, C of (14), p. 555.

The given PDE

$$u_{xx} - 2u_{xy} + u_{yy} = 0 \tag{P1}$$

is of the form (14) with

$$A = 1, \qquad 2B = 2, \qquad C = 1.$$

Step 2. Identify the type of the given PDE by using the table on p. 555 and determine whether $AC - B^2$ is positive, zero, or negative.

$$AC - B^2 = 1 \cdot 1 - 1^2 = 0.$$

Hence by the table on p. 555, **the given PDE is *parabolic.***

Step 3. Transform given PDE into normal form.

Using

$$Ay'^2 - 2By' + C = 0, \tag{15}$$

we obtain

$$y'^2 - 2y' + 1 = 0.$$

This can be factored:

$$(y' - 1)^2 = 0. \tag{A}$$

From the table on p. 556, we see that, for a parabolic equation, the new variables leading to the normal form are $v = x$ and $z = \Psi(x, y)$, where $\Psi = \text{const}$ is a solution $y = y(x)$ of (A). From (A) we obtain

$$y' - 1 = 0, \qquad y - x = \text{const}, \qquad z = x - y.$$

(Note that $z = y - x$ would do it equally well; try it.) Hence the new variables are

$$\begin{aligned} v &= x, \\ z &= x - y. \end{aligned} \tag{T}$$

We are following the method on p. 553 with our z playing the role of w on that page. The remainder of the work in Step 3 consists of transforming the partial derivatives that occur, into partial derivatives with respect to these new variables. This is done by the chain rule. For the first partial derivative of (T) with respect to x we obtain

$$\frac{\partial v}{\partial x} = v_x = 1, \qquad \frac{\partial z}{\partial x} = z_x = 1. \tag{T*}$$

From (T) and (T*), with the help of the chain rule (1), p. 393, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} \\ &= \frac{\partial u}{\partial v} + \frac{\partial u}{\partial z}, \end{aligned}$$

This can be written as

$$u_x = u_v + u_z.$$

Then

$$\begin{aligned} u_{xx} &= (u_v + u_z)_x \\ &= (u_v + u_z)_v v_x + (u_v + u_z)_z z_x && \text{(chain rule!)} \\ &= (u_{vv} + u_{zv})v_x + (u_{vz} + u_{zz})z_x \\ &= u_{vv} + u_{zv} + u_{vz} + u_{zz} && (v_x = 1,\ z_x = 1) \\ &= u_{vv} + 2u_{vz} + u_{zz}. \end{aligned}$$

(To test your understanding, can you write out these above equations in $\partial^2 u/\partial x^2$ notation (instead of the u_{xx} notation), using the chain rule for partials and see that you get the same result!)

Now we turn to partial differentiation with respect to y. The first partial derivative with respect to y is

$$\begin{aligned} u_y &= u_v v_y + u_z z_y \\ &= u_v \cdot 0 + u_z \cdot (-1) \qquad \text{(since } v_y = 0 \quad \text{and} \quad z_y = -1) \\ &= -u_z. \end{aligned}$$

We take the partial derivative of this with respect to x, obtaining

$$\begin{aligned} u_{yx} &= u_{xy} \\ &= (-u_z)_x \\ &= (-u_z)_v v_x + (-u_z)_z z_x \qquad \text{(chain rule)} \\ &= -u_{zv} \cdot 1 + (-u_{zz}) \cdot 1 \\ &= -u_{zv} - u_{zz} \\ &= -u_{vz} - u_{zz}. \end{aligned}$$

Finally, taking the partial derivative of u_y with respect to y gives

$$\begin{aligned} u_{yy} &= (-u_z)_y \\ &= (-u_z)_v v_y + (-u_z)_z\, z_y \qquad \text{(chain rule)} \\ &= -u_{zv} \cdot 0 + (-u_{zz}) \cdot (-1) \\ &= u_{zz}. \end{aligned}$$

We substitute all the partial derivatives just computed into the given PDE (P1)

$$\begin{aligned} u_{xx} + 2u_{xy} + u_{yy} &= u_{vv} \quad + 2u_{vz} \quad + u_{zz} \\ &\qquad\quad\; - 2u_{vz} \quad - 2u_{zz} \\ &\qquad\qquad\qquad\quad\; + u_{zz} \\ &= u_{vv} \\ &= 0 \qquad \text{[by right-hand side of (P1)].} \end{aligned}$$

This is the normal form of the parabolic equation.

Step 4. Solve the given PDE that you transformed into a normal form.

We now solve $u_{vv} = 0$ by two integrations. Using the method of Sec. 12.1, we note that u_{vv} can be treated like an ODE since it contains no z-derivatives. We get

$$u'' = 0$$

from which

$$\lambda^2 = 0, \qquad \lambda_1, \lambda_2 = 0 \qquad \text{(characteristic equation solved, Sec. 2.1)}$$

Case II $\qquad e^{-0}\,1, \quad e^{-0}\,x \qquad$ (table, p. 58)

$$u = c_1 + c_2 x.$$

Now back to the given PDE:

$$u(v, z) = c_1(z) + c_2(z)v$$

and recalling, from (T), what v and z are

$$u = c_1(x - y) + c_2(x - y)x.$$

Call $c_1 = f_2$ and $c_2 = f_1$ and get precisely the answer on p. A32.

Sec. 12.5 Modeling: Heat Flow from a Body in Space. Heat Equation

In a similar vein to Sec. 12.2, we derive the heat equation (3), p. 558 in Sec. 12.5. It models the temperature $u(x, y, z, t)$ of a (finite) body of homogeneous material in space.

Sec. 12.6 Heat Equation: Solution by Fourier Series. Steady Two-Dimensional Heat Problems. Dirichlet Problem

The first part of Sec. 12.6 models the one-dimensional heat equation (1), p. 559, and solves it by (9), p. 560, and (10) on the next page. This separation of variables parallels that for the wave equation, but, since the heat equation (1) involves u_t, whereas the wave equation involves u_{tt}, we get exponential functions in time in (9), p. 560, rather than cosine and sine in (12), p. 548, Sec. 12.3.

A single sine term as initial condition leads to a one-term solution (**Example 1**, p. 561, **Prob. 5**), whereas a more general initial condition leads to a Fourier series solution (**Example 3**, p. 562).

The second part of Sec. 12.6, pp. 564–566, models the two-dimensional time-independent heat problems by the Laplace equation (14), p. 564, which is solved by (17) and (18), p. 566.

Also check out the different types of boundary value problems on p. 564: (1) First BVP or Dirchlet problem (**Prob. 21**), (2) second BVP or Neumann problem, and (3) third BVP, mixed BVP, or Robin problem.

Problem Set 12.6. Page 566

5. Laterally insulated bar. For a better understanding of the solution, note the following: The solution to this problem shares some similarities with that of Prob. 5 of Sec. 12.3 on p. 237 of this Manual. There are two reasons for this:

- The one-dimensional heat equation (1), p. 559, is quite similar to the one-dimensional wave equation (1), p. 545. Their solutions, described by (9) and (10), pp. 560–561, for the heat equation and (12), (14), and (15), pp. 548 and 549, for the wave equation, share some similarities, the foremost being that both require the computing of coefficients of Fourier sine series.
- In both our problems, the initial condition $u(x, 0)$ (meaning initial temperature for the heat equation, initial deflection for the wave equation, respectively) consists only of *single* sine terms, and *hence we do not need to compute the coefficients of the Fourier sine series*. (For **Prob. 7** we would need to compute these coefficients as required by (10)).

From (9), p. 560, we have

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = \sin(0.1\pi x).$$

We are given that $L = 10$ so that, by comparison,

$$\sin(0.1\pi x) = \sin\left(\frac{\pi x}{10}\right) = \sin\left(\frac{n\pi x}{L}\right)$$

from which we see that $n = 1$. This means that the initial condition is such that the solution is given by the first eigenfunction and, just like in Prob. 5 of Sec. 12.3, we do not have to compute anything for (10). Indeed,

$$B_1 = 1, \qquad B_2 = 0, \qquad B_3 = 0, \text{ etc.}$$

This reduces (9), p. 560, to

$$u(x, t) = B_1 \sin \frac{n\pi x}{L} e^{-\lambda_1^2 t}. \tag{9*}$$

We also need

$$c^2 = \frac{K}{\sigma\rho} = \frac{1.04\left[\frac{\text{cal}}{\text{cm}\cdot\text{sec}\cdot{}^\circ\text{C}}\right]}{0.056\left[\frac{\text{cal}}{\text{g}\cdot{}^\circ\text{C}}\right]\cdot 10.6\left[\frac{\text{g}}{\text{cm}^3}\right]} = 1.75202\left[\frac{\text{cm}^2}{\text{sec}}\right],$$

where K is thermal conductivity, σ is specific heat, and ρ is density. Now

$$\lambda_1 = \frac{c\cdot 1\cdot \pi}{L}$$

so that

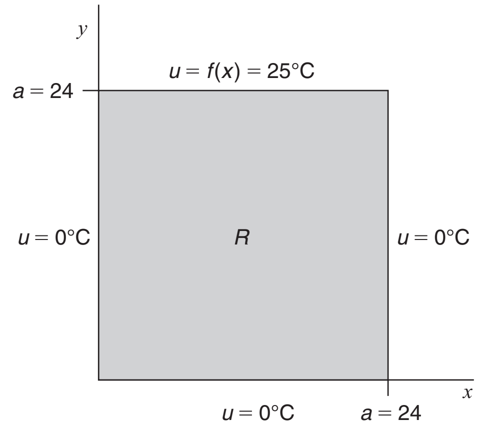
$$\lambda_1^2 = \frac{c^2\cdot 1^2\cdot \pi^2}{L^2} = \frac{1.75202\cdot 1\cdot (3.14159)^2\left[\frac{\text{cm}^2}{\text{sec}}\right]}{10^2\left[\text{cm}^2\right]} = 0.172917[\text{sec}^{-1}].$$

We have computed all the parts for (9*) and get the answer

$$\begin{aligned} u(x,t) &= B_1 \sin\frac{n\pi x}{L} e^{-\lambda_1^2 t} \\ &= 1\cdot\sin(0.1\pi x)\exp(-1.75202\cdot 1\cdot \pi^2 t/100) = \sin(0.1\pi x)\cdot e^{-0.172917t}\left[{}^\circ\text{C}\right], \end{aligned}$$

which corresponds to the answer on p. A32 by taking one step further and noting that $\pi^2 1.75202/100 = 0.172917$.

21. Heat flow in a plate. Dirichlet problem. *Setting up the problem.* We put the given boundary conditions into Fig. 297, p. 567, of the textbook, and obtain the following diagram for a thin plate:



Sec. 12.6 Prob. 21. Square plate R with given boundary values

Solution. The problem is a ***Dirichlet problem*** because u is prescribed (given) on the boundary, as explained on p. 546. The solution to this steady two-dimensional heat problem is modeled on pp. 564–566 of the textbook, and you should take a look at it before we continue.

Our problem corresponds to the one in the text with $f(x) = 25 = \text{const}$. Hence we obtain the solution of our problem from (17) and (18), p. 566, with $a = b = 24$. We begin with (18), which for this problem takes the form

$$\begin{aligned} A_n^* &= \frac{2}{24\sinh n\pi}\int_0^{24} 25\sin\left(\frac{n\pi x}{24}\right)dx \\ &= \frac{50}{24\sinh n\pi}\int_0^{24}\sin\left(\frac{n\pi x}{24}\right)dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{50}{24 \sinh n\pi}\left(-\frac{24}{n\pi}\right)\cos\frac{n\pi x}{24}\bigg|_{x=0}^{24} \\
&= -\frac{50}{n\pi \sinh n\pi}(\cos n\pi - 1) \\
&= -\frac{50}{n\pi \sinh n\pi}[(-1)^n - 1] \\
&= +\frac{100}{n\pi \sinh n\pi}, \qquad\qquad \text{for } n = 1, 3, 5, \cdots
\end{aligned}
$$

and 0 for even n. We substitute this into (17), p. 566, and obtain the series

$$u = \frac{100}{\pi}\sum_{n \text{ odd}} \frac{1}{n \sinh n\pi} \sin\frac{n\pi x}{24} \sinh\frac{n\pi y}{24}.$$

Note that we sum only over odd n because $A_n^* = 0$ for even n. If, in this answer, we write $2n - 1$ instead of n, then we automatically have the desired summation and can drop the condition "n odd." This is the form of the answer given on p. A32 in App. 2 of the textbook.

Sec. 12.7 Heat Equation: Modeling Very Long Bars. Solution by Fourier Integrals and Transforms

In this section we return to the heat equation in one dimension.

In the previous section the x-interval was finite. Now it is *infinite*. This changes the method of solution from Fourier series to Fourier integrals on pp. 569–570, after the PDE has been separated on p. 568.

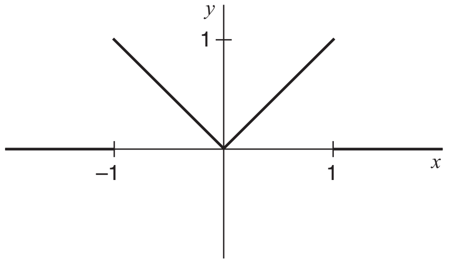
Example 1, p. 570, gives an application. **Examples 2** and **3**, pp. 571–573, solve the same model by operational methods.

Examples 1–3 concern a bar (a very long wire) that extends to infinity in both directions (that is, from $-\infty$ to ∞). **Example 4**, p. 573, shows what happens if the bar extends to ∞ in one direction only, so that we can use the Fourier sine transform.

Problem Set 12.7. Page 574

5. Modeling heat problems in very long bars. Use of Fourier integrals. We are given that the initial condition for our problem is

$$u(x, 0) = f(x) = |x| \quad \text{if} \quad |x| < 1, \qquad 0 \text{ otherwise.}$$



Sec. 12.7 Prob. 5. Initial condition

Since $f(-x) = f(x), f$ is even. Hence in (8), p. 569, $B(p) = 0$. We have to compute $A(p)$, as reasoned on p. 515 of Sec. 11.7 of the textbook. By (8),

$$
\begin{aligned}
A(p) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos pv\,dv \\
&= \frac{1}{\pi}\int_{-1}^{0} -v\cos pv\,dv + \frac{1}{\pi}\int_{0}^{1} v\cos pv\,dv \\
&= \frac{1}{\pi}\int_{0}^{-1} v\cos pv\,dv + \frac{1}{\pi}\int_{0}^{1} v\cos pv\,dv.
\end{aligned}
$$

Integration by parts (verify by tabular integration as in Prob. 7, Sec. 12.3, p. 238 of this Manual) gives

$$\int v\cos pv\,dv = \frac{pv\sin pv + \cos pv}{p^2} + \text{const.}$$

Then

$$\int_0^{-1} v\cos pv\,dv = \frac{p(-1)\sin(-p) + \cos(-p)}{p^2} - \frac{\cos 0}{p^2} = \frac{p\sin p + \cos p - 1}{p^2}.$$

Similarly,

$$\int_0^{1} v\cos pv\,dv = \frac{p\sin p + \cos p - 1}{p^2}.$$

Putting them together

$$A(p) = \frac{1}{\pi}\frac{p\sin p + \cos p - 1}{p^2} + \frac{1}{\pi}\frac{p\sin p + \cos p - 1}{p^2} = \frac{2}{\pi}\frac{p\sin p + \cos p - 1}{p^2}.$$

Hence by (6), p. 569, we obtain the desired answer as the Fourier integral:

$$u(x,t) = \int_0^1 \frac{2}{\pi}\,\frac{p\sin p + \cos p - 1}{p^2}\,e^{-c^2p^2t}\,dp.$$

Remark. We could have used symmetry (that is, that the initial condition is an even function and that the area under the curve to the left of the origin is equal to the area under the curve to the right of the origin) and thus written

$$A(p) = \frac{2}{\pi}\int_0^1 v\cos pv\,dv$$

and shortened the solution. But you do not need to always come up with the most elegant solution to solve a problem! We shall use the more elegant approach of symmetry in Prob. 7.

7. Modeling heat problems in very long bars. Use of Fourier integrals. We note that $h_1(x) = \sin x$ is an odd function (p. 486). $h_2(x) = 1/x$ is an odd function. Hence

$$f(x) = h_1(x)\cdot h_2(x) = \frac{\sin x}{x} = u(x,0)$$

is an even function, being the product of two odd functions (verify!). Hence $B(p)$ in (8), p. 569, is zero, as reasoned on p. 515 of Sec. 11.7.

For $A(p)$ we obtain, from (8),

$$
\begin{aligned}
A(p) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(p)\,\cos pv\,dv \\
&= \frac{1}{\pi}\int_{-1}^{1} \frac{\sin v}{v}\,\cos pv\,dv \qquad \text{(IV)} \\
&= \frac{2}{\pi}\int_{0}^{1} \frac{\sin v}{v}\,\cos pv\,dv.
\end{aligned}
$$

This is precisely integral (7) on p. 514 of Sec. 11.7, also know as the **Dirichlet discontinuous factor.** Its value is $\pi/2$ for $0 < p < 1$ and 0 for $p > 1$. Hence multiplication by $2/\pi$—the factor in (IV)—gives the values $A(p) = 1$ for $0 < p < 1$ and $A(p) = 0$ for $p > 1$. (The value at $p = 1$ is $(\pi/4)(2/\pi) = \frac{1}{2}$; this is of no interest here because we are concerned with integral (6).)

Substituting $A(p)$, just obtained, and $B(p) = 0$ from the start of the problem into (6), p. 569, gives us

$$
\begin{aligned}
u(x,t) &= \int_0^{\infty} [A(p)\cos px + B(p)\sin px]\, e^{-c^2p^2t}\,dp \\
&= \int_0^{1} \cos(px)\, e^{-c^2p^2t}\,dp.
\end{aligned}
$$

Sec. 12.8 Modeling: Membrane, Two-Dimensional Wave Equation

Here we are modeling a vibrating membrane, such as a drumhead (see Fig. 301, p. 576), and get the two-dimensional wave equation (3), p. 577. Compare this section with Sec. 12.2 (pp. 543–445), the analogy of the one-dimensional case of the vibrating string, where the physical ideas are given in greater detail.

Good modeling is the art of disregarding minor factors. Disregarding is done to get a model that is still faithful enough but also simple enough so that it can be solved. Can you find passages in the present derivation where we disregarded minor factors?

Sec. 12.9 Rectangular Membrane. Double Fourier Series

We set up the two-dimensional wave equation (1) with boundary condition (2), and initial conditions (3a) and (3b), p. 577, and solve the problem for a ***rectangular*** **membrane** R in Fig. 302, p. 577. Pay close attention to the steps in solving this problem, as similar steps will be used in Sec. 12.10, where we solve the more difficult problem of a circular membrane. For the rectangular membrane we proceed as follows:

Step 1, pp. 578–579. We make two separations of variables to get three ODEs involving the three independent variables x, y (rectangular coordinates), and time t.

Step 2, pp. 579–580, Example 1, p. 581. We find infinitely many solutions satisfying the boundary condition "membrane fixed along the boundary" (a rectangle). We call these solutions "eigenfunctions" of the problem.

Step 3, pp. 582–583, Example 2, pp. 583–584. We solve the whole problem by Fourier series.

Problem Set 12.9. Page 584

5. **Coefficients of a double Fourier series** can be obtained following the idea in the text. For $f(x, y) = y$, in the square $0 < x < 1$, $0 < y < 1$, the calculations are as follows. (Here we use the formula numbers of the text.) The desired series is obtained from (15) with $a = b = 1$ in the form

$$
\begin{aligned}
f(x, y) = y &= \sum \left(\sum B_{mn} \sin m\pi x \sin n\pi y \right) \\
&= \sum K_m(y) \sin m\pi x \quad \text{(sum over } m) \qquad (15)
\end{aligned}
$$

where the notation

$$K_m(y) = \sum B_{mn} \sin n\pi y \quad \text{(sum over } n) \qquad (16)$$

was used. Now fix y. Then the second line of (15) is the Fourier sine series of $f(x, y) = y$ considered as a function of x (hence as a constant, but this is not essential). Thus, by (4) in Sec. 11.3, its Fourier coefficients are

$$b_m = K_m(y) = 2 \int_0^1 y \sin m\pi x \, dx. \qquad (17)$$

We can pull out y from under the integral sign (since we integrate with respect to x) and integrate, obtaining

$$
\begin{aligned}
K_m(y) &= \frac{2y}{m\pi}(-\cos m\pi + 1) \\
&= \frac{2y}{m\pi}[-(-1)^m + 1] \\
&= \frac{4y}{m\pi} \qquad \text{if } m \text{ is odd and } 0 \text{ for even } m
\end{aligned}
$$

(because $(-1)^m = 1$ for even m). By (6**) in Sec. 11.3 (with y instead of x and $L = 1$) the coefficients of the Fourier series of the function $K_m(y)$ just obtained are

$$
\begin{aligned}
B_{mn} &= 2 \int_0^1 K_m(y) \sin n\pi y \, dy \\
&= 2 \int_0^1 \frac{4 \, y \sin n\pi y}{m\pi} \, dy \\
&= \frac{8}{m\pi} \int_0^1 y \sin n\pi y \, dy.
\end{aligned}
$$

Integration by parts gives

$$B_{mn} = \frac{8}{nm\pi^2} \left(-y \cos n\pi y \Big|_{y=0}^{1} + \int_0^1 \cos n\pi y \, dy \right).$$

The integral gives a sine, which is zero at $y = 0$ and $y = n\pi$. The integral-free part is zero at the lower limit. At the upper limit it gives

$$\frac{8}{nm\pi^2}(-(-1)^n) = \frac{(-1)^{n+1} 8}{nm\pi^2}.$$

Remember that this is the expression when m is odd, whereas for even m these coefficients are zero.

Sec. 12.10 Laplacian in Polar Coordinates. Circular Membrane. Fourier–Bessel Series

Here we continue our modeling of vibrating membranes that we started in Sec. 12.9, p. 577. Here R is a ***circular*** **membrane** (a drumhead) as shown in Fig. 307, p. 586.

We transform the two-dimensional wave equation, given by PDE (1), p. 585, into suitable coordinates for which the boundary of the domain, in which the PDE is to be solved, has a simple representation. Here these are polar coordinates, and the transformation gives (5), p. 586. Further steps are:

Step 1, p. 587. Apply separation of variables to (7), obtaining a Bessel ODE (12) for the radial coordinate $r = s/k$ and the ODE (10) for the time coordinate t.

Step 2, pp. 587–589. Find infinitely many Bessel functions $J_0(k_m r)$ satisfying the boundary condition that the membrane is fixed on the boundary circle $r = R$; see (15).

Step 3, p. 589, Example 1, p. 590. A Fourier–Bessel series (17) with coefficients (19) will give us the solution of the whole problem (7)–(9), defined on p. 586, with initial velocity $g(r) = 0$ in (9b).

Problem Set l2.10. Page 591

3. Alternate form of Laplacian in polar coordinates. We want to show that (5), p. 586, that is,

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{5}$$

can be written as (5′)

$$\nabla^2 u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}. \tag{5′}$$

We proceed as follows. By the product rule we get that

$$\begin{aligned}(ru_r)_r &= (r)_r \cdot u_r + r \cdot (u_r)_r \\ &= 1 \cdot u_r + r \cdot u_{rr}.\end{aligned}$$

Hence, multiplying both sides by $1/r$

$$\frac{1}{r}(ru_r)_r = \frac{1}{r} \cdot u_r + u_{rr}.$$

Now

$$\begin{aligned}\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} &= \frac{1}{r}\cdot u_r + u_{rr} + \frac{1}{r^2}u_{\theta\theta} && \text{(substituting the last equation)}\\ &= u_{rr} + \frac{1}{r}\cdot u_r + \frac{1}{r^2}u_{\theta\theta} && \text{(rearranging terms)}\\ &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} && \text{(equivalent notation …)}\\ &= \nabla^2 u && \text{(… which is the right-hand side of (5)).}\end{aligned}$$

This shows that (5) and (5′) are equivalent.

5. Electrostatic potential in a disk. Use of formula (20), p. 591. We are given that the boundary values are

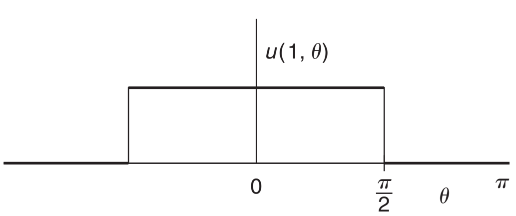
$$u(1, \theta) = f(\theta) = 220 \quad \text{if} \quad -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \qquad \text{and} \qquad 0 \qquad \text{otherwise.}$$

We sketch as shown below. We note that the period p of $f(\theta)$ is 2π, which is reasonable as we are dealing with a disk $r < R = 1$. Hence the period $p = 2L = 2\pi$ so that $L = \pi$. Furthermore, $f(\theta)$ is an even function. Hence we use (6*), p. 486, to compute the coefficients of the Fourier series as required by (20), p. 591. Since $f(\theta)$ is even, the coefficients b_n in (20) are 0, that is, $f(\theta)$ is not represented by any sine terms but only cosine terms.

We compute

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi/2} f(\theta)\,d\theta \qquad \left[\text{since } f(\theta) = 0,\ \theta \geq \frac{\pi}{2}\right] \\
&= \frac{1}{\pi}\int_0^{\pi/2} 220\,d\theta \\
&= \frac{220}{\pi}\big[\theta\big]_0^{\pi/2} \\
&= \frac{220}{\pi}\cdot\frac{\pi}{2} = 110. \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \\
&= \frac{2}{L}\int_0^{\pi/2} f(\theta)\cos\frac{n\pi\theta}{L}\,d\theta \\
&= \frac{2}{\pi}\int_0^{\pi/2} f(\theta)\cos n\theta\,d\theta \\
&= \frac{2}{\pi}\int_0^{\pi/2} 220\cdot\cos\, n\theta\,d\theta = \frac{2}{\pi}\cdot 220\int_0^{\pi/2}\cos\, n\theta\,d\theta \\
&= \frac{440}{\pi}\left[\frac{\sin n\theta}{n}\right]_0^{\pi/2} = \frac{440}{n\pi}\sin\frac{n\pi}{2}.
\end{aligned}$$

For even n this is 0. For $n = 1, 5, 9, \cdots$ this equals $440/(n\pi)$, and for $n = 3, 7, 11, \cdots$ it equals $-440/(n\pi)$. Writing the Fourier series out gives the answer shown on p. A33.



Sec. 12.10 Prob. 5. Boundary potential

11. Semidisk. The idea sketched in the answer on p. A34 is conceived by symmetry. Proceeding in that way will guarantee that, on the horizontal axis, we have potential 0. This can be confirmed by noting that the sine terms are all 0 when $\theta = 0$ or π, the horizontal axis. See also Example 2, p. 485. You may want to write down the details.

13. Circular membrane. Frequency and tension. We want to know how doubling the tension T of a drum affects the frequency f_m of an eigenfunction u_m. Our intuition and experience of listening to drums tells us that the frequency must also go up. The mathematics of this section confirms it and tells us by how much.

From p. 588, the vibration of a drum corresponding to u_m (given by (17)) has the frequency $f_m = \lambda_m/2\pi$ cycles per unit time. Hence

$$f_m \propto \lambda_m, \tag{A}$$

where λ_m is the eigenvalue of the eigenfunction u_m.

By (6), p. 586,

$$c^2 = \frac{T}{p}$$

so that the tension T is

$$T = c^2 p,$$

where p is the density of the drum. Furthermore,

$$\lambda_m = \frac{c\alpha_m}{R} \qquad \text{(p. 588).}$$

Hence

$$c = \frac{\lambda_m}{\alpha_m} R.$$

Thus

$$T = \left(\frac{\lambda_m}{\alpha_m} R\right)^2 p = \left(p\frac{R^2}{\alpha_m^2}\right)\lambda_m^2.$$

This means that

$$T \propto \lambda_m^2$$

so that

$$\lambda_m \propto \sqrt{T}. \tag{B}$$

(A) and (B) give

$$f_m \propto \sqrt{T}.$$

Thus if we increase the tension of the drum by a factor of 2, the frequency of the eigenfunction only increases by a factor of $\sqrt{2}$.

Sec. 12.11 Laplace's Equation in Cylindrical and Spherical Coordinates. Potential

Cylindrical coordinates are obtained from polar coordinates by adding the z-axis, so that (5) for $\nabla^2 u$ (see p. 594) follows immediately from (5) on p. 586 in Sec. 12.10. Really new, and not so immediate, is the transformation of $\nabla^2 u$ to the form (7), p. 594, in spherical coordinates. Separation in these coordinates leads to an Euler ODE and to Legendre's ODE. This accounts for the importance of the latter and the corresponding development (17), p. 596, in terms of Legendre polynomials, called a **Fourier–Legendre series**. The name indicates that its coefficients are obtained by orthogonality, just as the Fourier coefficients; see pp. 504–507.

Problem 9 shows how your knowledge of ODEs comes up all the time—here it requires solving an Euler–Cauchy equation (1), p. 71 in Sec. 2.5.

Problem Set 12.11. Page 598

9. Potentials depending only on *r*. Spherical symmetry. We show that the only solution of Laplace's equation (1), depending only on $r = \sqrt{x^2 + y^2 + z^2}$, is

$$\text{(S)} \qquad u = \frac{c}{r} + k, \qquad \text{where } c, k \text{ const.}$$

We use spherical coordinates. In that setting, we have by (7), p. 594, and (1) that

$$\text{(A)} \qquad \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin\phi}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Now, for (A), u depends on r, θ, ϕ, i.e., $u = u(r, \theta, \phi)$. In our problem u *depends only on* r, that is, $u = u(r)$. Thus in (A)

$$\frac{\partial u}{\partial \theta} = 0, \qquad \frac{\partial u}{\partial \phi} = 0,$$

and hence

$$\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad \frac{\partial^2 u}{\partial \phi^2} = 0.$$

Thus (A) simplifies to

$$\text{(B)} \qquad \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = 0.$$

However, since u depends only on r, the partials are ordinary derivatives, that is,

$$\frac{\partial^2 u}{\partial r^2} = \frac{d^2 u}{dr^2}, \qquad \frac{\partial u}{\partial r} = \frac{du}{dr}.$$

Hence (B) is

$$\text{(B}'\text{)} \qquad \nabla^2 u = \frac{d^2 u}{dr^2} + \frac{2}{r}\frac{du}{dr} = 0.$$

We have to solve (B′). We can write it in a simpler notation:

$$u'' + \frac{2}{r}u' = 0.$$

Multiplying both sides by r^2 gives

$$\text{(B}''\text{)} \qquad r^2 u'' + 2ru' = 0.$$

But this is precisely an Euler–Cauchy equation of the form (1), p. 71 in Sec. 2.5 (in that notation!)

$$x^2y'' + 2xy' + 0y = 0. \tag{C}$$

(*Notation switching:* Here x, y are the regular notations for ODEs in Part A of the text.) We solve (C) as given. The auxiliary equation (with $a = 2$) is

$$m^2 + (a-1)m + b = m^2 + m + 0 = 0.$$

The equation $m^2 + m = 0$ is $m(m-1) = 0$ so that

$$m_1 = -1, \qquad m_2 = 0.$$

Hence by (4), p. 71, the solution to (C) is

$$\begin{aligned} y &= c_1x^{m_1} + c_2x^{m_2} \\ &= c_1x^{-1} + c_2x^0 \\ &= c_1x^{-1} + c_2. \end{aligned}$$

Back to our original notation. Thus (B″) has the solution

$$u = \widetilde{c}_1 r^{-1} + \widetilde{c}_2,$$

which is precisely (S) with $\widetilde{c}_1 = c$ and $\widetilde{c}_2 = k$.

Remark. Note that the solution on p. A34 of the textbook uses *separation of variables* instead of identifying the resulting ODE as an Euler–Cauchy type.

13. Potential between two spheres. The region is bounded by two concentric spheres, which are kept at constant potentials. Hence the potential between the spheres will be spherically symmetric, that is, the equipotential surfaces will be concentric spheres. Now a spherically symmetric solution of the three-dimensional Laplace equation is

$$u = u(r) = \frac{c}{r} + k, \qquad \text{where } c, k \text{ constant,}$$

as we have just shown in Prob. 9. The constants c and k can be determined by the two boundary conditions $u(2) = 220$ and $u(4) = 140$. Thus

$$u(2) = \frac{c}{2} + k = 220. \tag{D}$$

Furthermore,

$$u(4) = \frac{c}{4} + k = 140,$$

or, multiplied by 2,

$$2u(4) = \frac{c}{2} + 2k = 280. \tag{E}$$

Subtracting equation (E) from (D) gives

$$k = 60.$$

From this and (D) we have

$$\frac{c}{2} = 160, \qquad \text{hence} \qquad c = 320.$$

Hence the solution is

$$u(r) = \frac{320}{r} + 60.$$

You should check our result, sketch it, and compare it to that of Prob. 12.

19. Boundary value problems in spherical coordinates. Special Fourier–Legendre series. These series were introduced in Example 1 of Sec. 11.6, pp. 505–506, of the textbook. They are of the form

$$a_0P_0 + a_1P_1 + a_2P_2 + \cdots ,$$

where P_0, P_1, P_2 are the Legendre polynomials as defined by (11), p. 178. Since x is one of our coordinates in space, we must choose another notation. We choose w and use ϕ obtained by setting $w = \cos\phi$. The Legendre polynomials $P_n(w)$ involve powers of w: thus $P_n(\cos\phi)$ involves powers of $\cos\phi$. Accordingly, we have to transform $\cos 2\phi$ into powers of $\cos\phi$.

From (10), p. A64 of Part A3.1 of App. 3 of the textbook, we have

$$\cos^2\theta = \tfrac{1}{2}(1 - \cos 2\theta).$$

Solving this for $\cos 2\theta$ we have

$$\tfrac{1}{2}\cos 2\theta = \cos^2\theta - \tfrac{1}{2}.$$

Multiply by 2

$$\text{(E)} \qquad \cos 2\theta = 2\cos^2\theta - 1 = 2w^2 - 1.$$

This must be expressed in Legendre polynomials. The Legendre polynomials are, by (11′), p. 178:

$$P_0 = 1, \qquad P_1 = x, \qquad P_2 = \tfrac{1}{2}(3x^2 - 1).$$

Since our expression (E) involves powers of 0 and 2, we need P_0 and P_2. Thus we set

$$\text{(F)} \qquad 2w^2 - 1 = A \cdot P_0(w) + B \cdot P_2(w).$$

Hence

$$2w^2 - 1 = A + \tfrac{3}{2}Bw^2 - \tfrac{1}{2}B.$$

Comparing coefficients we have

$$\begin{aligned} &[w^0] \qquad & A - \tfrac{1}{2}B &= -1, \\ &[w^2] \qquad & \tfrac{3}{2}B &= 2. \end{aligned}$$

The second equation gives

$$B = 2 \cdot \tfrac{2}{3} = \tfrac{4}{3},$$

which we substitute into the first equation to get

$$A = -1 + \tfrac{1}{2} \cdot \tfrac{4}{3} = -\tfrac{1}{3}.$$

We put the values for the constants A and B, just obtained, into (F) and get the result

$$\text{(G)} \qquad 2w^2 - 1 = -\tfrac{1}{3} \cdot P_0(w) + \tfrac{4}{3} \cdot P_2(w).$$

We also need (16a), p. 596, which is

$$\text{(16a)} \qquad u_n(r, \phi) = A_n r^n P_n(\cos\,\phi).$$

From (G) and (16a) we see that the answer is

$$u(r, \phi) = -\tfrac{1}{3} \cdot P_0(\cos\phi) + \tfrac{4}{3} r^2 \cdot P_2(\cos\phi).$$

Remark. Be aware that, in the present case, the coefficient formulas (19) or (19*), p. 596, were not needed because the boundary condition was so simple that the coefficients were already known to us. Note further that $P_0 = 1 = \text{const}$, but our notations $P_0(w)$ and $P_0(\cos\phi)$ are correct because a constant is a special case of a function of any variable.

Sec. 12.12 Solution of PDEs by Laplace Transforms

We conclude this chapter on ODEs by showing that Laplace transforms (Chap. 6, pp. 203–253) can also be used quite elegantly to solve PDEs, as explained in **Example 1**, pp. 600–602, and **Prob. 5.** We hope to have conveyed to you the incredible richness and power of PDEs in modeling important problems from engineering and physics.

Problem Set 12.12. Page 602

5. First-order differential equation. The boundary conditions mean that $w(x, t)$ vanishes on the positive parts of the coordinate axes in the xt-plane. Let W be the Laplace transform of $w(x, t)$ considered as a function of t; write $W = W(x,\, s)$. The derivative w_t has the transform sW because $w(x, 0) = 0$. The transform of t on the right is $1/s^2$. Hence we first have

$$x\,W_x + s\,W = \frac{x}{s^2}.$$

This is a first-order linear ODE with the independent variable x. Division by x gives

$$W_x + \frac{s\,W}{x} = \frac{1}{s^2}.$$

Its solution is given by the integral formula (4), p. 28 in Sec. 1.5 of the textbook. Using the notation in that section, we obtain

$$p = \frac{s}{x}, \qquad h = \int p\,dx = s \ln\,x, \qquad e^h = x^s, \qquad e^{-h} = \frac{1}{x^s}.$$

Hence (4) in Sec. 1.5 with the "constant" of integration depending on s, gives, since $1/s^2$ does not depend on x,

$$W(x,\, s) = \frac{1}{x^s}\left[\int \frac{x^s}{s^2}\,dx + c(s)\right] = \frac{c(s)}{x^s} + \frac{x}{s^2\,(s+1)}$$

(note that x^s cancels in the second term, leaving the factor x). Here we must have $c(s) = 0$ for $W(x, s)$ to be finite at $x = 0$. Then

$$W(x, s) = \frac{x}{s^2\,(s+1)}.$$

Now

$$\frac{1}{s^2\,(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}.$$

This has the inverse Laplace transform $t - 1 + e^{-t}$ and gives the solution $w(x, t) = x\,(t - 1 + e^{-t})$. You should work out the details of the steps by identifying the techniques of Laplace transforms used (see Chap. 6, pp. 203–253 of the text and this Manual pp. 79–106).